Action Potential Propagation in Branched Cable Network

Adv Topics Math Physiology: Modeling Neuronal Dynamics

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Overview

- **Equations: Cable Equations (PDE) and Hodgkin Huxley Kinetics**
- **Construction of tree data structure**
- **Implement algorithm that solves tridiagonal system on the tree**
- **Experiment to see whether propagation fails or succeeds at a junction (all or none)**
- **Can we get differential propagation with this model ?**

Cable Equations with Hodgkin Huxley Kinetics:

Interior Equations:

$$C\frac{\partial v}{\partial t} + g(v-E) = \frac{r}{2\rho}\frac{\partial^2 v}{\partial x^2}$$

$$g = g_{Na} + g_K + g_L$$

$$E = \frac{g_{Na}E_{Na} + g_K E_L + g_L E_L}{g_{Na} + g_K + g_L}$$

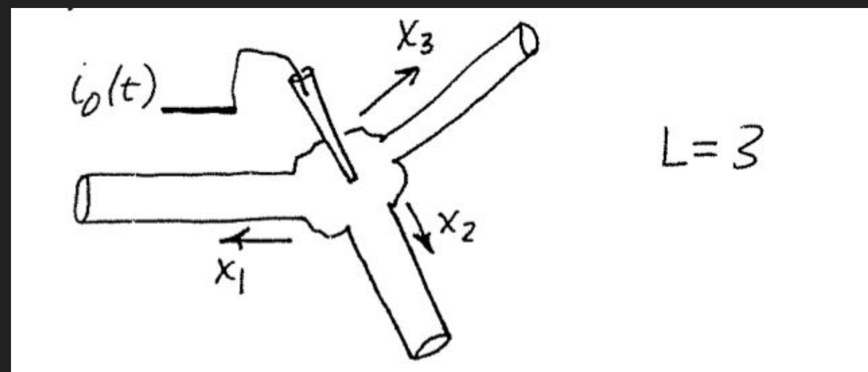
$$g_{Na} = \overline{g_{Na}}\, m^3 h$$

$$g_K = \overline{g_K}\, n^4$$

$$g_L = \overline{g_L}$$

$$\frac{\partial s}{\partial t} = \alpha_s(v)(1-s) - \beta_s(v)s$$

where $s = m, h,$ or n



Boundary Condition: Voltage is continuous and current adds up

$$v_0(t) = v_1(0,t) = v_2(0,t) = \cdots = v_L(0,t)$$

$$i_0(t) = A_0\left(C\frac{\partial v_0}{\partial t} + g_0(v_0 - E_0)\right) + \sum_{l=1}^{L}\left(-\frac{\pi r_l^2}{\rho}\right)\frac{\partial v_l}{\partial x_l}(0,t)$$

Numerical Method: Crank-Nicolson Method

$$\frac{S_j^{k+1/2} - S_j^{k-1/2}}{\Delta t} = \alpha_s(v_j^k)\left(1 - \frac{S_j^{k+1/2} + S_j^{k-1/2}}{2}\right)$$

$$- \beta_s(v_j^k)\left(\frac{S_j^{k+1/2} + S_j^{k-1/2}}{2}\right)$$

$$c\,\frac{v_j^{k+1} - v_j^k}{\Delta t} + g_j^{k+1/2}\left(\frac{v_j^{k+1} + v_j^k}{2} - E_j^{k+1/2}\right)$$

$$= \frac{r}{2\rho}\left(D^+D^- \frac{v^{k+1} + v^k}{2}\right)_j$$

$$i_0^{k+1/2} = \tilde{A}_0\left(c\,\frac{v_0^{k+1} - v_0^k}{\Delta t} + \tilde{g}_0^{k+1/2}\left(\frac{v_0^{k+1} + v_0^k}{2} - \tilde{E}_0^{k+1/2}\right)\right)$$

$$+ \sum_{l=1}^{L}\left(-\frac{\pi r_l^2}{\rho}\right) \frac{\left(\frac{v_{l1}^{k+1} + v_{l1}^k}{2}\right) - \left(\frac{v_0^{k+1} + v_0^k}{2}\right)}{\Delta x_l}$$

$$\tilde{A}_0 = A_0 + \sum_{l=1}^{L} \frac{1}{2}\left(2\pi r_l \Delta x_l\right)$$

$$\tilde{g}_0 = \frac{A_0 g_0 + \sum_{l=1}^{L} \frac{1}{2}\left(2\pi r_l \Delta x_l\right) g_{l0}}{\tilde{A}_0}$$

$$\tilde{E}_0 = \frac{A_0 g_0 E_0 + \sum_{l=1}^{L} \frac{1}{2}(2\pi r_l)\Delta x_l\, g_{l0} E_{l0}}{\tilde{A}_0 \tilde{g}_0}$$

Construction of Tree data structure:

- Arbitrary connected graph of nodes with no loops.
- Number the nodes 1,...,N and arbitrarily select one to be the root R.
- Since G is a connected tree, it is possible to find a unique chain between 1 and any node i.
- R(i) is the root-ward neighbor of node i. Note R(1) = 0.
- fc(i) = j is the leafward neighbor of node i with the smallest index (first child)
- If i has multiple leaf-ward neighbors (e.g. j,p,q) , then these are denoted as “next sibling” of the “child” before it : ns(j) = p , ns(p) = q , ns(q) = 0 if its the last sibling.

$$-a_i \cdot V_{R(i)} + b_i V_i - \sum_{j \in \mathcal{L}(i)} c_j V_j = W_j$$

Simple tree example:

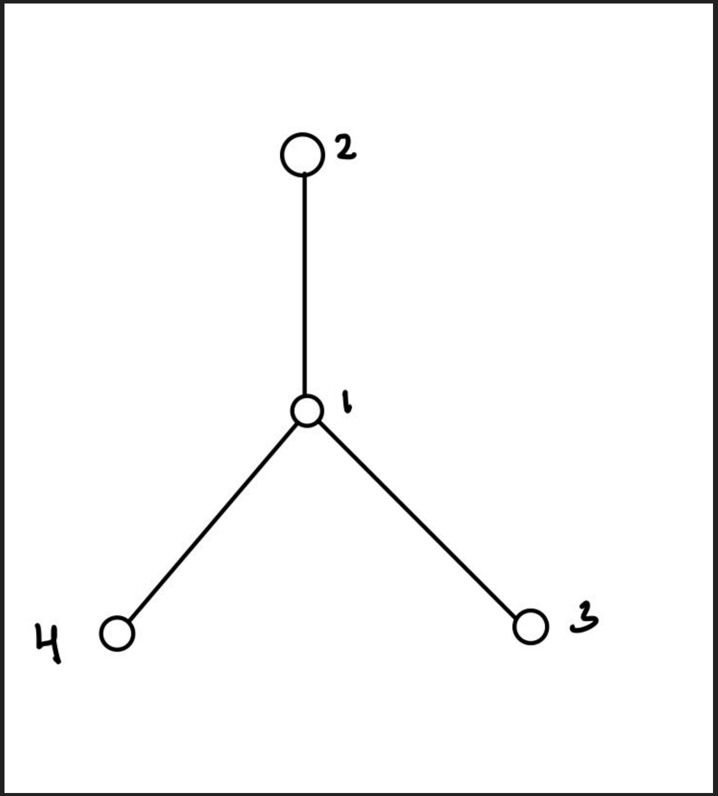
fc(1) = 2 ns(1) = 0 R(1) = 0

fc(2) = 0 ns(2) = 1 R(2) = 1

fc(3) = 0 ns(3) = 4 R(3) = 1

fc(4) = 0 ns(4) = 0 R(4) = 1

This is a high-level description of the basic tree tree will run simulations on, however, more nodes will be added along the branches.



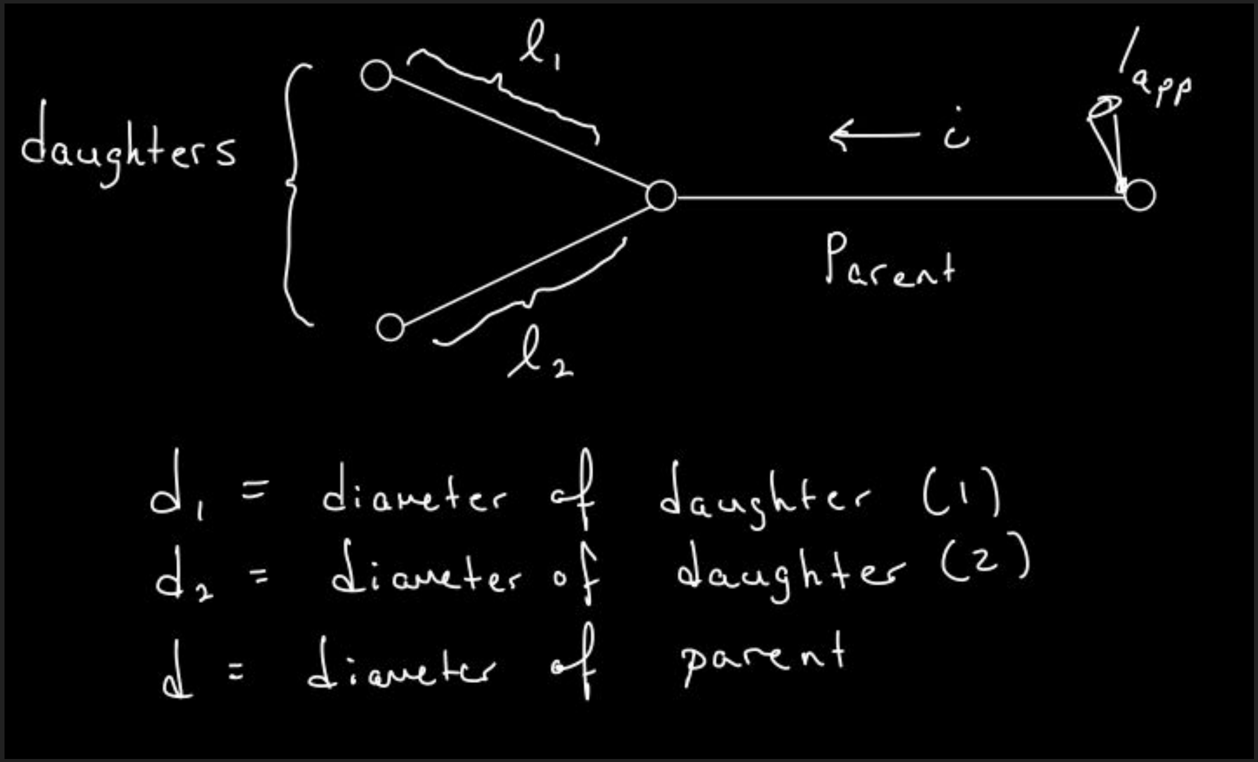
Tridiagonal Solver Along the Tree:

```
function v = vnew(a, b, c, W, R, fc, ns, J)
v=W;

for i=J:(-1):1
    j=fc(i);
    while(j~=0)
        b(i)=b(i)-c(j)*a(j);
        v(i)=v(i)+c(j)*v(j);
        j=ns(j);
    end
    a(i)=a(i)/b(i);
    v(i)=v(i)/b(i);
end
for i=2:(+1):J
    v(i) = v(i) + a(i)*v(R(i));
end
```

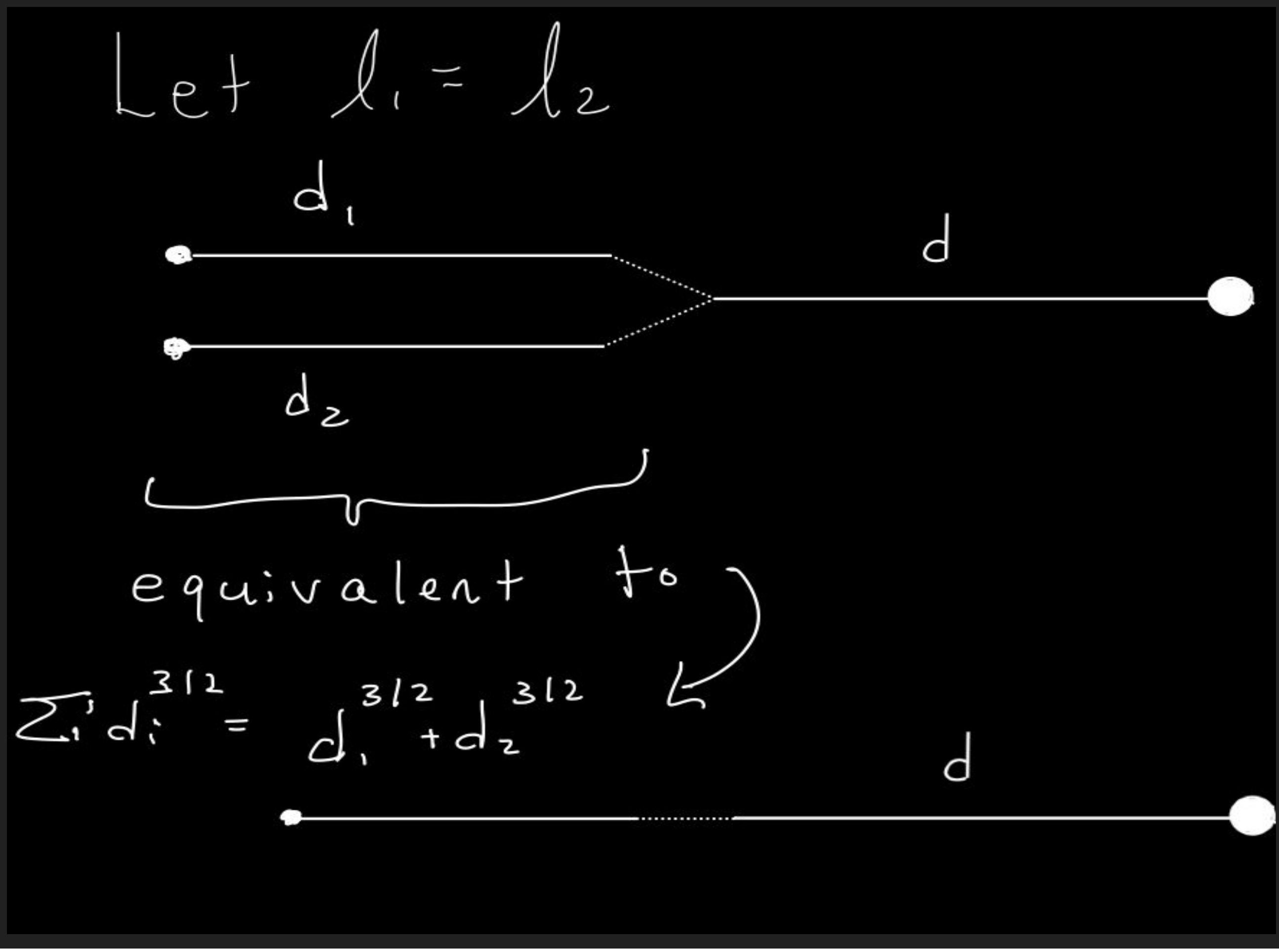
Colliding AP's

Colliding action potentials cancel each other out because the refractory period of either spike prevents the continuation of an impulse in either direction:

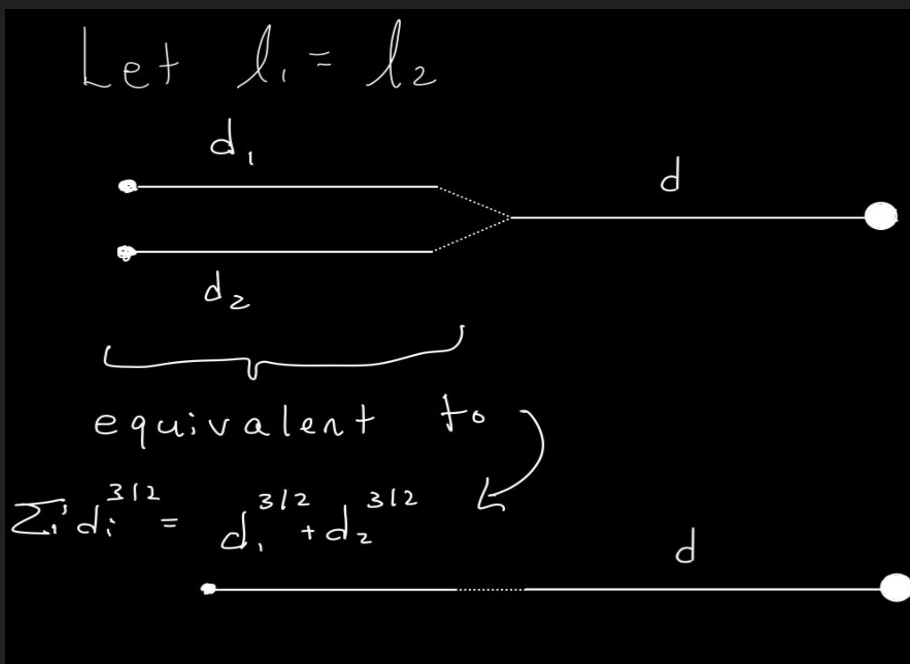
Consider a parent leading into two daughters:



Rall's Equivalent Cylinder (1962)



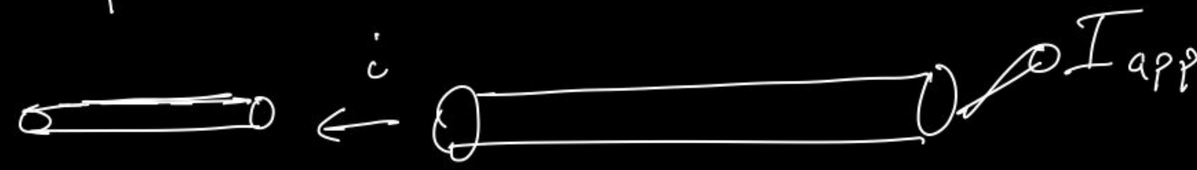
Rall’s Equivalent Cylinder (1962)

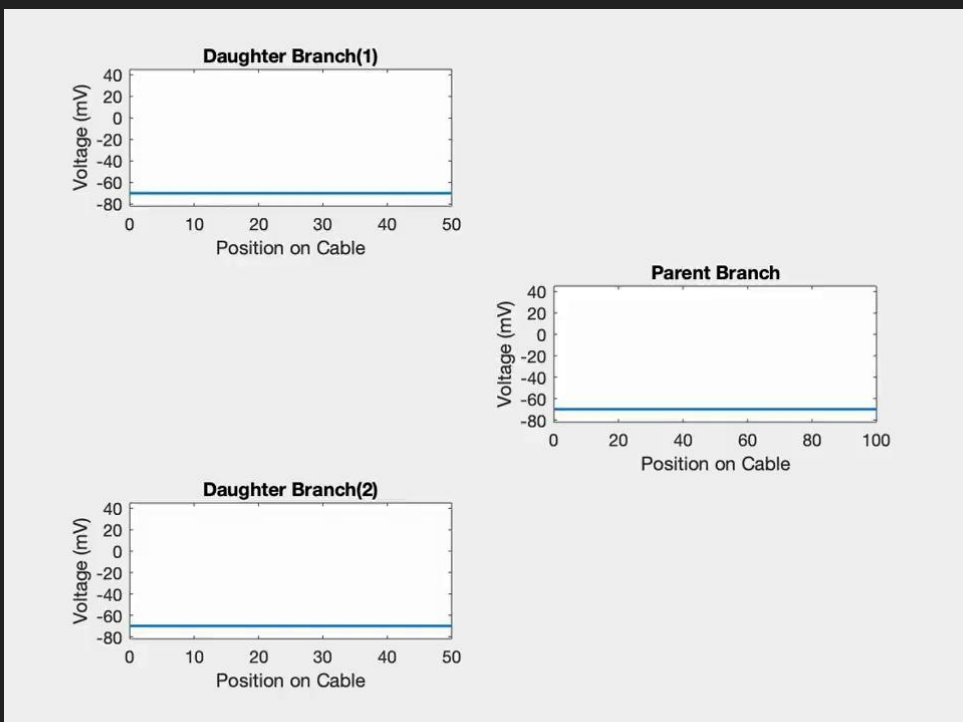


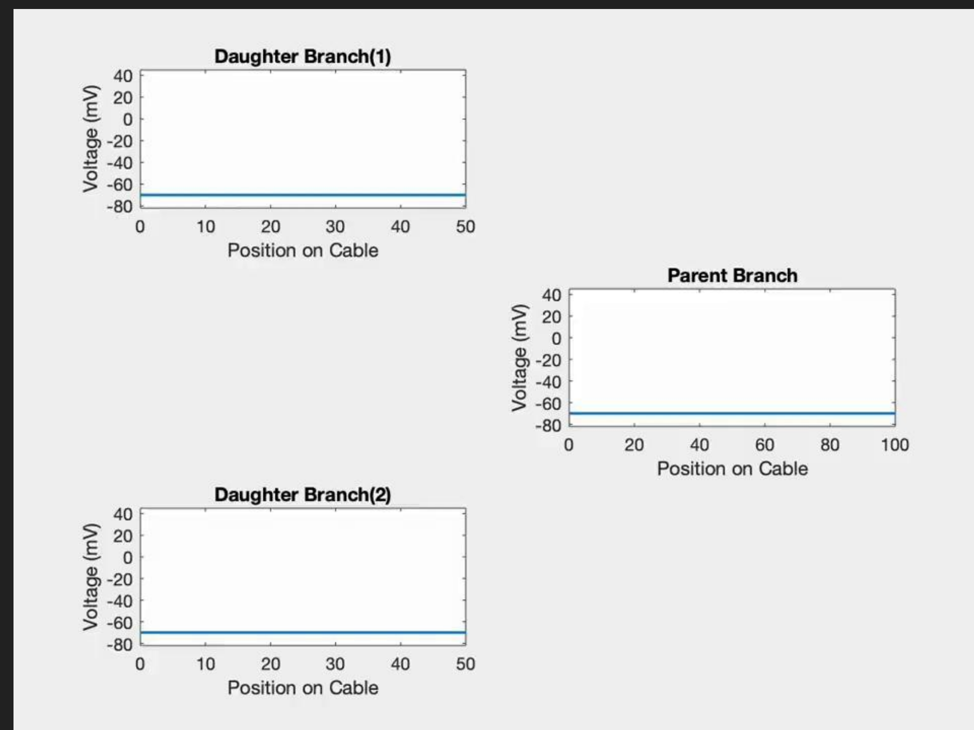
- Since the electrotonic lengths of the daughters is the the same, we can reduce the branched case.
- This should ensure that propagation will follow “all or none” behavior.

Case 1:

If $d^{3/2} >> \sum_i d_i^{3/2}$:





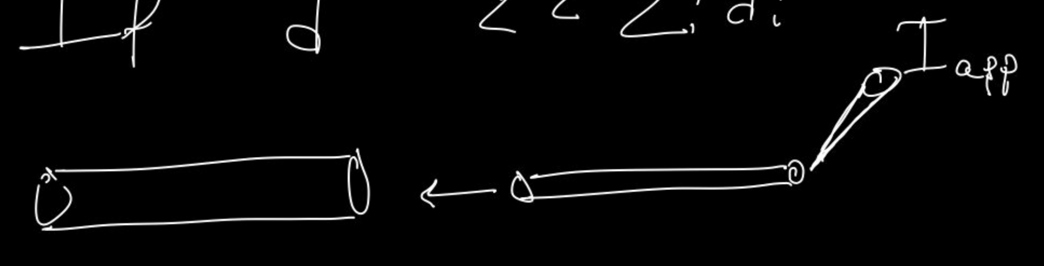


Check that solutions on each branch are the same :

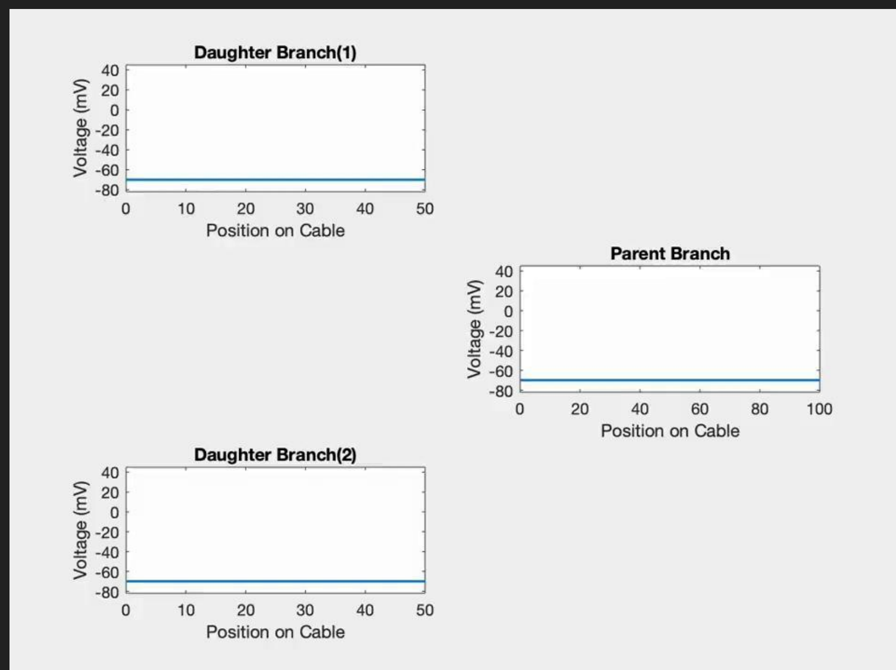
```
>> v(401:600)-v(201:400)

ans =

  Columns 1 through 23

     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0

  Columns 24 through 46

     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0

  Columns 47 through 69

     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0

  Columns 70 through 92

     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0

  Columns 93 through 115

     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0

  Columns 116 through 138

     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0

  Columns 139 through 161

     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0

  Columns 162 through 184

     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0

  Columns 185 through 200

     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0
```

Case 2:

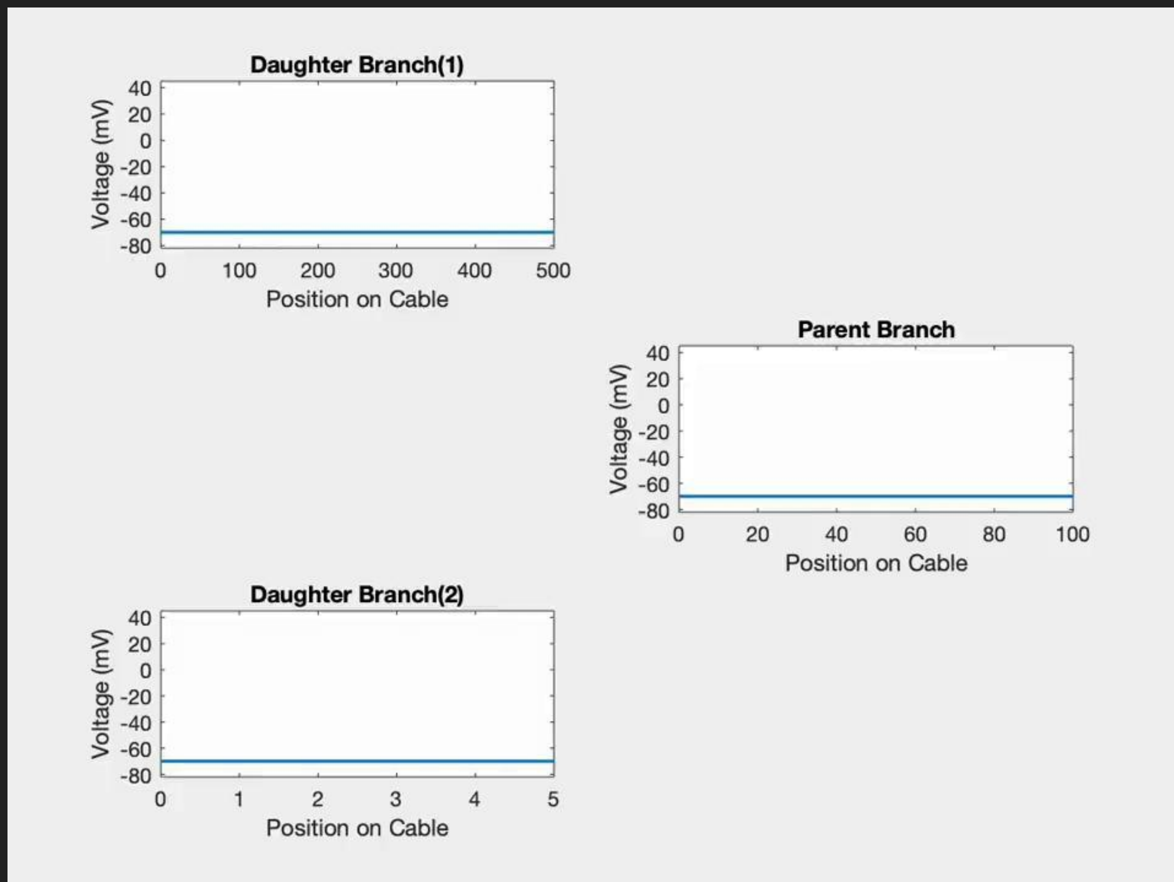
If $d^{3/2} \ll \sum_i d_i^{3/2}$



Too much membrane area to cover… propagation will FAIL



Differential Propagation: Electrotonic Lengths Unequal



References

[1] Peskin, Charles S. “Mathematical Aspects of Neurophysiology .” Hodgkin Huxley Equations. 2000.

[2] Rinzel J, Rall W. Transient response in a dendritic neuron model for current injected at one branch. *Biophys J*. 1974;14(10):759-790. doi:10.1016/S0006-3495(74)85948-5

Link to the google slides for access to the videos:

https://docs.google.com/presentation/d/1nVkldiOWmzgTDwWfFoqgfYqK1C7cdr1Iv82MetPq1-4/edit?usp=sharing