Computational Cardiology Lecture 1

Circulatory models and parameter identifiability

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Fall 2024

States and observations

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The state determines how the model continues. The observation map determines what the experiment can distinguish.

Resolution follows the observable

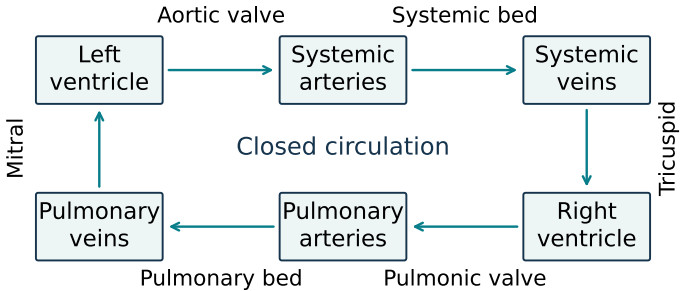
$$\begin{aligned}\text{mean perfusion} &\longrightarrow \overline{Q},\\ \text{valve timing} &\longrightarrow P_i(t),\ q_{ij}(t),\\ \text{wall mechanics} &\longrightarrow \sigma(x,t).\end{aligned}$$

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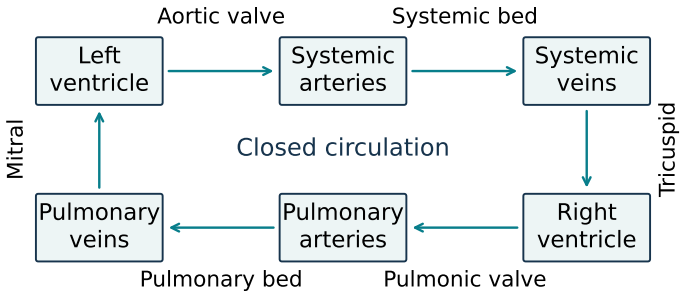
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A compartment volume does not contain a spatial stress distribution.

The two-pump circuit



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The closed network contains two ventricles and four vascular storage regions. Atrial storage is not a separate state in this grouping.

The incidence matrix

For each directed edge, put -1 at its donor and $+1$ at its receiver.

$$\dot{\boldsymbol{V}} = B\boldsymbol{q}.$$

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$$\dot{\boldsymbol{V}} = B\boldsymbol{q}.$$

Every column sums to zero: $\mathbf{1}^{\mathsf{T}} B = 0$.

Conservation of total volume

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With an external source s, the right-hand side becomes $\mathbf{1}^\mathsf{T} s$. The boundary determines the conserved quantity.

Flow and compliant storage

$$q_{ij} = \frac{P_i - P_j}{R_{ij}}, \qquad V_i = V_{0i} + C_i(P_i - P_{\mathrm{ext},i}).$$

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Fluid pressure difference drives flow. Transmural pressure determines storage.

Dimensions of the constitutive laws

$$[R] = \frac{\mathrm{mmHg\,min}}{\mathrm{L}}, \qquad [C] = \frac{\mathrm{L}}{\mathrm{mmHg}}.$$

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$$[RC] = \mathrm{min}.$$

A pressure gauge shift preserves storage only when the exterior reference shifts consistently.

Transport over repeated beats

For n identical beats in T minutes,

$$\overline{Q} = \frac{n\,SV}{1000T} = \frac{F\,SV}{1000}.$$

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For unequal beats, replace $n\,SV$ by $\sum_k SV_k$.

A mean-flow calculation

$$F = 80\ \text{min}^{-1}, \qquad SV = 75\ \text{mL}, \qquad \Delta P = 90\ \text{mmHg}.$$

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$$\overline{Q} = 6\ \text{L/min}, \qquad R_{\text{eff}} = \frac{90}{6} = 15\ \text{mmHg min/L}.$$

An exact parameter symmetry

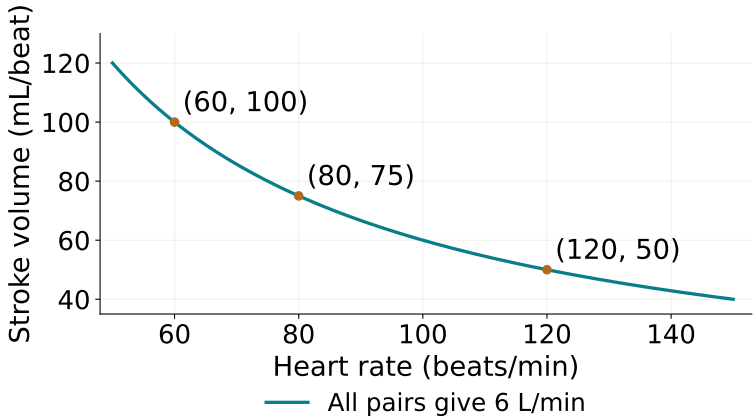
$$H(F, SV) = \frac{F\,SV}{1000}, \qquad H(aF, SV/a) = H(F, SV), \quad a > 0.$$

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The same mean flow corresponds to infinitely many positive parameter pairs.

One mean-flow observation leaves many possibilities



A level set of the observation map

$$SV = \frac{1000 Q_0}{F}.$$

The pairs $(60, 100)$, $(80, 75)$ and $(120, 50)$ all lie on the $Q_0 = 6$ level set.

The unobservable perturbation

$$\delta Q = \frac{SV\,\delta F + F\,\delta SV}{1000}.$$

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$$DH \begin{pmatrix} F \\ -SV \end{pmatrix} = 0, \qquad \frac{\delta F}{F} = -\frac{\delta SV}{SV}.$$

Opposite relative changes cancel to first order.

Adding an independent measurement

$$\widetilde{H}(F, SV) = \begin{pmatrix} F & SV/1000 \\ & F \end{pmatrix}.$$

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$$\widetilde{H}(F, SV) = \begin{pmatrix} F\, SV/1000 \\ F \end{pmatrix}.$$

$$\det D\widetilde{H} = -\frac{F}{1000} \neq 0 \quad (F > 0).$$

Measured rate removes this particular ambiguity.

Uncertainty in inferred stroke volume

$$SV = \frac{1000Q}{F}.$$

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$$\frac{\delta SV}{SV} \simeq \frac{\delta Q}{Q} - \frac{\delta F}{F}.$$

Identifiability supplies a unique local inverse; measurement precision determines how uncertain that inverse remains.

Information retained by the model

Mean flow and measured rate determine stroke volume under the repeated-cycle assumption.

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Mean flow and measured rate determine stroke volume under the repeated-cycle assumption.
They do not determine a ventricular pressure waveform, valve leakage or a spatial stress field. Those quantities require additional states and observations.