Computational Cardiology Lecture 2

A periodically forced circulation

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Stored volumes as state variables

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$$\dot{x} = Bq, \qquad P = D_C(\phi)^{-1} x.$$

Pressure requires both stored volume and phase.

A closed six-region network

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Prescribed cardiac phase

$$\phi(t_s) = \phi(0) + F t_s/60, \qquad T_b = 60/F.$$

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At $F = 80\ \mathrm{min}^{-1}$, $T_b = 0.75$ s. Both ventricles use the same clock.

The hydraulic time increment

$$h = \frac{0.001875}{60} = 3.125 \times 10^{-5} \text{ min.}$$

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$$Fh = \frac{1}{400} \text{ cycle.}$$

Flow in L/min must be multiplied by a step in minutes.

The initial conserved volume

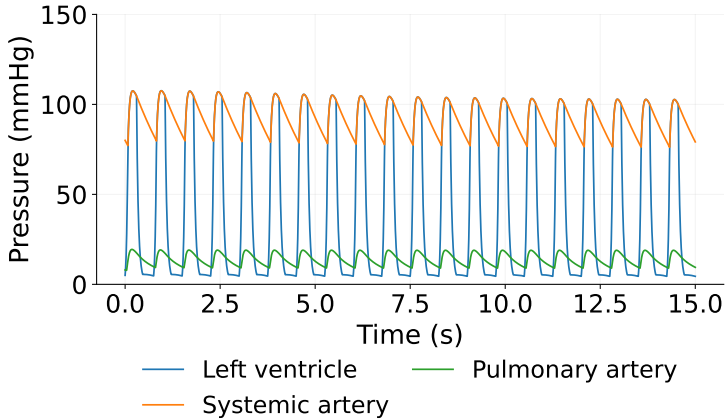
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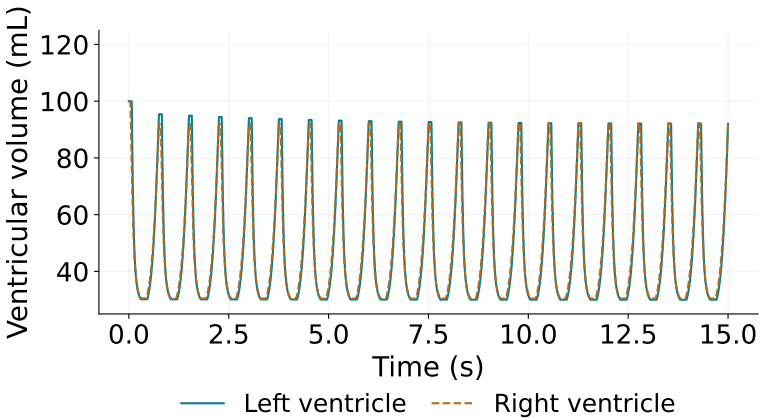
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Ventricular compliances at phase zero are 0.0146 and 0.0365 L/mmHg.

Pressure (mmHg)



Ventricular volume (mL)



A periodically forced state

Pressures and compartment volumes change throughout a beat. Total volume remains constant.

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$$P(t + T_b) \simeq P(t)$$

describes periodicity, not a stationary pressure.

The one-beat return map

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All comparisons use the same starting phase and initialized total volume.

Phase-aligned comparison

$$r_{\mathrm{per}}(t) = \max_i |P_i(t + T_b) - P_i(t)|.$$

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A small return-map residual measures the remaining transient. It does not measure time-step error.

Sum the compartment equations

$$\dot{V}_i = (Bq)_i.$$

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$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_i V_i = \mathbf{1}^{\mathsf{T}} B q = 0.$$

Every internal transfer occurs once with each sign.

The same invariant after a time step

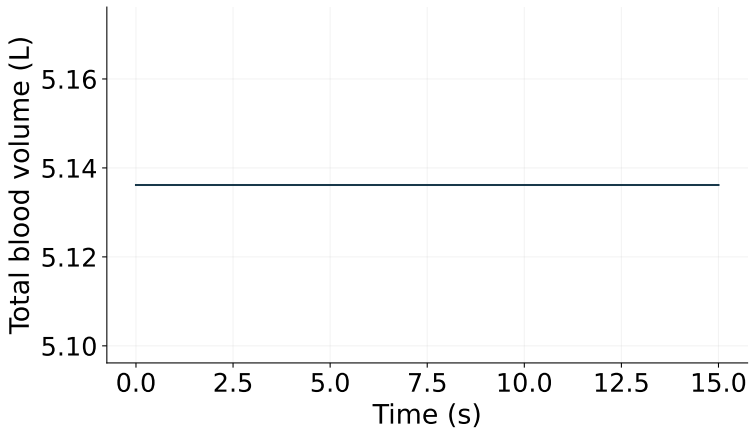
$$D_C^{n+1} P^{n+1} - D_C^n P^n = hBq^{n+1}.$$

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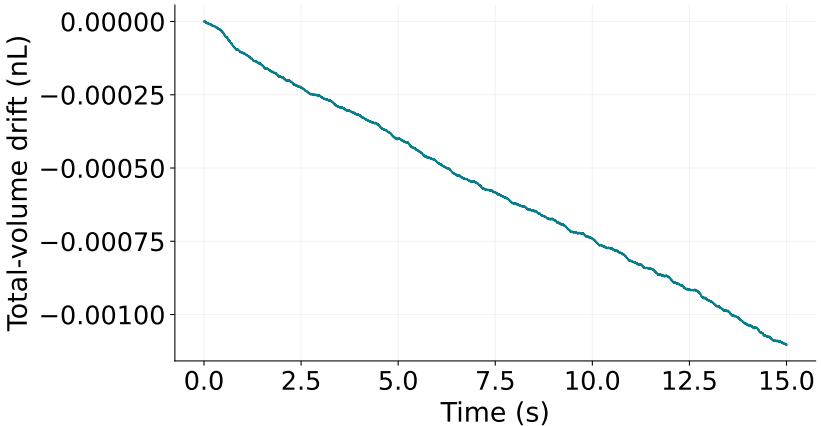
$$D_C^{n+1} P^{n+1} - D_C^n P^n = hBq^{n+1}.$$

Left multiplication by $\mathbf{1}^\mathsf{T}$ removes every internal flow.

Total blood volume (L)



Total-volume drift (nL)



Reconstruct volume at the saved phase

$$\varepsilon_V(t) = \sum_i [V_{d,i} + C_i(\phi(t))P_i(t)] - V_{\text{tot}}(0).$$

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A fixed phase-zero compliance would measure a different quantity.

The error in a mismatched diagnostic

$$V_{\text{wrong}}(t) - V_{\text{true}}(t) = \sum_i [C_i(0) - C_i(t)] P_i(t).$$

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This term can be large without any loss of blood from the closed model.

Three distinct residuals

$$
\begin{aligned}
r_{\mathrm{per}} &= \|P_h(t+T_b) - P_h(t)\|_\infty, \\
r_V &= |V_{\mathrm{tot},h}(t) - V_{\mathrm{tot}}(0)|, \\
d_h &= \|P_h(t) - P_{h/2}(t)\|_\infty.
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They measure periodicity, conservation and refinement sensitivity.

Resistance changes the dynamics

$$q_s = \frac{P_{sa} - P_{sv}}{R_s}.$$

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Changing R_s changes redistribution while leaving $\mathbf{1}^{\mathsf{T}} B = 0$. Conservation does not require identical trajectories.

Implicit coupling

New compliances and new pressures enter the hydraulic balance together.

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New compliances and new pressures enter the hydraulic balance together. The pressure solve must also satisfy the ideal-valve inequalities; an algebraically conserved state can still have inconsistent valve labels.