Computational Cardiology Lecture 3

Implicit pressure equations and ideal-valve complementarity

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The stored quantity

$$V = V_d + CP, \qquad \dot{V} = C\dot{P} + \dot{C}P.$$

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Omitting $\dot{C}P$ removes pressure generation by changing compliance.

An isolated contracting compartment

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$$C^{n+1} = C^n/2 \quad \Longrightarrow \quad P^{n+1} = 2P^n.$$

Discretize stressed volume

$$C_i^{n+1} P_i^{n+1} - C_i^n P_i^n = h(Bq^{n+1})_i.$$

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Old and new compliance enter at their corresponding time levels.

One active connection

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It contributes $hg \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ to the two pressure rows.

The pressure matrix

$$(D_C^{n+1} + hBGB^{\mathsf{T}})P^{n+1} = D_C^n P^n.$$

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Active valves contribute $g = 1/R$; closed valves contribute zero.

A unique fixed-state solve

For $z \neq 0$,

$$z^{\mathsf{T}} A z = \sum_i C_i z_i^2 + h \sum_e g_e (z_i - z_j)^2 > 0.$$

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Positive compliances make A positive definite. Valve consistency is a separate condition.

The ideal-valve law

$$q = \frac{\max(d, 0)}{R}, \qquad d = P_{\text{up}} - P_{\text{down}}.$$

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$$q \geq 0, \quad Rq - d \geq 0, \quad q(Rq - d) = 0.$$

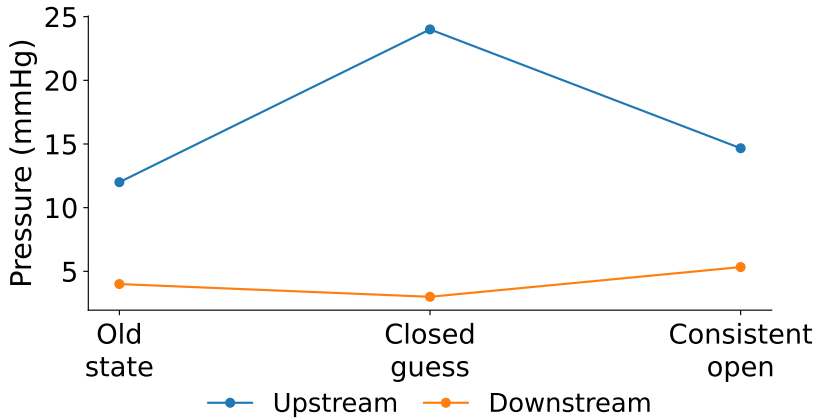
Active-set iteration

Choose valve labels, solve the coupled pressure system, then evaluate new pressure differences.

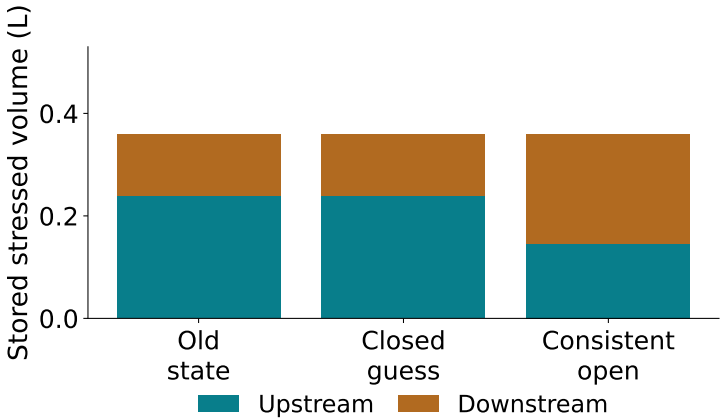
Active-set iteration

Choose valve labels, solve the coupled pressure system, then evaluate new pressure differences. An inconsistent label changes the matrix and requires another solve.

A guessed valve state must be checked



Conservation alone cannot choose the valve



A rejected closed candidate

Old stressed volumes are $(.24, .12)$ L; new compliances are $(.01, .04)$ L/mmHg.

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$$P_{\text{closed}} = (24, 3) \text{ mmHg}.$$

Total volume is correct, but $24 - 3 > 0$ contradicts closure.

The accepted open candidate

$$\begin{pmatrix} .02 & -.01 \\ -.01 & .05 \end{pmatrix} P = \begin{pmatrix} .24 \\ .12 \end{pmatrix}.$$

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$$P = (44/3, 16/3), \qquad q = 14/3 \text{ L/min}.$$

One transfer, two balance equations

$$hq = \frac{7}{75}\ \mathrm{L}, \qquad V^{n+1} = \left(\frac{11}{75}, \frac{16}{75}\right) \mathrm{L}.$$

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The loss from one compartment equals the gain by the other.

Constant-compliance relaxation

For a forward pressure difference $d(0) \geq 0$,

$$d = P_1 - P_2, \qquad \dot{d} = -\lambda d, \qquad \lambda = \frac{1/C_1 + 1/C_2}{R}.$$

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$$d(t) = d(0)e^{-\lambda t}.$$

The limiting pressure

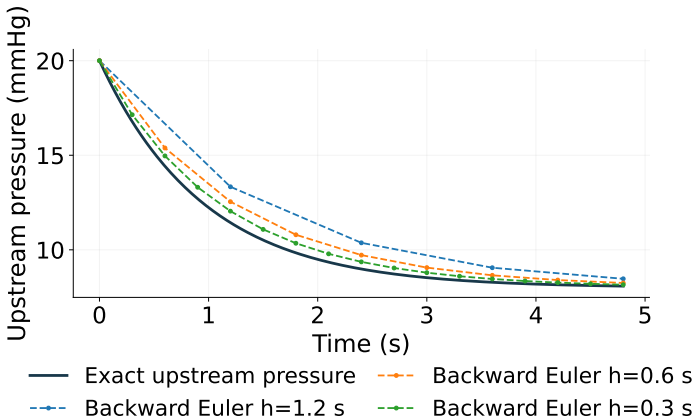
$$P_\infty = \frac{C_1 P_1(0) + C_2 P_2(0)}{C_1 + C_2}.$$

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With $C = (.01, .04)$ and $P(0) = (20, 5)$, $P_\infty = 8$ mmHg.

An implicit conservative update still has time error



Backward Euler and the exact factor

$$d_{n+1} = \frac{d_n}{1 + h\lambda}, \qquad d(t+h) = e^{-h\lambda} d(t).$$

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Stability does not make the two factors equal.

Conservation and admissibility

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A step is accepted only when the storage equations and all valve inequalities hold together. Constant-compliance relaxation supplies an exact reference; switching and changing compliance require their own error analysis.