Computational Cardiology Lecture 6

Fourier-mode error analysis of a transport discretization

Zan Ahmad

Fall 2024

A continuum transport mode

$$c(x,t) = 1 + Ae^{-Dk^2 t}\sin(k(x-at)), \qquad k = 2\pi/L.$$

A continuum transport mode

$$c(x,t) = 1 + Ae^{-Dk^2 t}\sin(k(x-at)), \qquad k = 2\pi/L.$$

Advection changes phase; diffusion changes amplitude.

Differentiate the reference

$$\partial_t c = -a\partial_x c + D\partial_{xx} c.$$

Differentiate the reference

$$\partial_t c = -a\partial_x c + D\partial_{xx} c.$$

The sine satisfies periodic boundary conditions and retains mean one.

Match the represented state

$$\bar{c}_i = 1 + A\,\mathrm{sinc}(k\Delta x/2)e^{-Dk^2t}\sin(k(x_i - at)).$$

Match the represented state

$$\bar{c}_i = 1 + A\,\mathrm{sinc}(k\Delta x/2)e^{-Dk^2t}\sin(k(x_i - at)).$$

Here $\mathrm{sinc}\, z = \sin z/z$. Cell averages differ from midpoint samples.

An RMS error

$$E_h = \left(\frac{1}{N} \sum_i |\bar{c}_i^{\text{num}} - \bar{c}_i^{\text{ref}}|^2 \right)^{1/2}.$$

An RMS error

$$E_h = \left(\frac{1}{N} \sum_i |\bar{c}_i^{\mathrm{num}} - \bar{c}_i^{\mathrm{ref}}|^2 \right)^{1/2}.$$

All values refer to the same physical time and the same averaging convention.

Observed order

$$E_h \simeq C h^p \quad \Longrightarrow \quad p_{\text{obs}} = \log_2(E_h / E_{h/2}).$$

Observed order

$$E_h \simeq C h^p \quad \Longrightarrow \quad p_{\mathrm{obs}} = \log_2(E_h / E_{h/2}).$$

A slope requires an asymptotic regime; two nearly equal runs are not sufficient.

Fourier diagonalization

$$c_i = z e^{\mathrm{i}i\theta}, \qquad c_{i\pm1} = e^{\pm\mathrm{i}\theta} c_i.$$

Fourier diagonalization

$$c_i = z e^{\mathrm{i} i\theta}, \qquad c_{i\pm 1} = e^{\pm \mathrm{i}\theta} c_i.$$

A translation-invariant stencil acts on this vector by multiplication.

The spatial eigenvalue

$$\lambda_h = -\frac{|a|}{\Delta x}(1-\cos\theta) - \frac{4D}{\Delta x^2}\sin^2(\theta/2) - \mathrm{i}\frac{a}{\Delta x}\sin\theta.$$

The spatial eigenvalue

$$\lambda_h = -\frac{|a|}{\Delta x}(1-\cos\theta) - \frac{4D}{\Delta x^2}\sin^2(\theta/2) - \mathrm{i}\frac{a}{\Delta x}\sin\theta.$$

Its real part damps the mode; its imaginary part transports its phase.

The leading spatial error

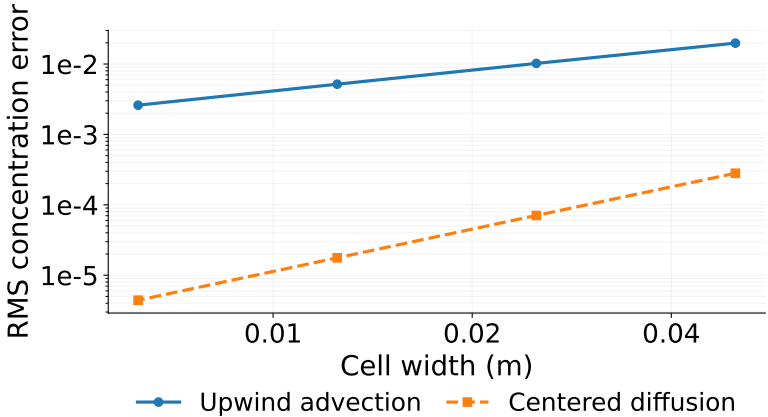
$$\lambda_h = -\mathrm{i}ak - Dk^2 - \frac{|a|k^2}{2}\Delta x + O(\Delta x^2).$$

The leading spatial error

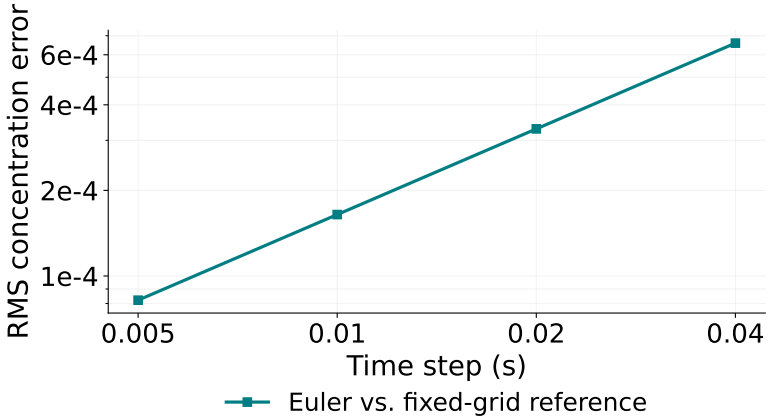
$$\lambda_h = -\mathrm{i}ak - Dk^2 - \frac{|a|k^2}{2}\Delta x + O(\Delta x^2).$$

Upwind advection is first order. Pure centered diffusion removes the $O(\Delta x)$ term.

Space: time error removed



Time: Euler at fixed grid



Exact semi-discrete evolution

$$z(t) = e^{\lambda_h t} z(0).$$

Exact semi-discrete evolution

$$z(t) = e^{\lambda_h t} z(0).$$

Compare this with the continuum mode to isolate spatial error.

Euler amplification at a fixed grid

$$z^{n+1} = (1 + \Delta t \lambda_h) z^n.$$

Euler amplification at a fixed grid

$$z^{n+1} = (1 + \Delta t \lambda_h) z^n.$$

Compare $(1 + \Delta t \lambda_h)^{T/\Delta t}$ with $e^{T\lambda_h}$ to isolate time error.

Accumulated temporal error

$$\frac{T}{\Delta t}\log(1+\Delta t\lambda_h) = T\lambda_h - \frac{T\Delta t}{2}\lambda_h^2 + O(\Delta t^2).$$

Accumulated temporal error

$$\frac{T}{\Delta t}\log(1+\Delta t\lambda_h) = T\lambda_h - \frac{T\Delta t}{2}\lambda_h^2 + O(\Delta t^2).$$

The leading difference is first order in Δt.

A grid-scale perturbation

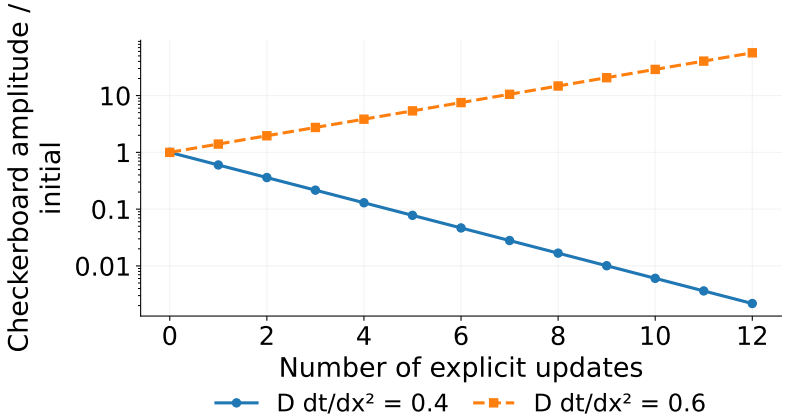
$$c_i = 1 + \varepsilon(-1)^i, \qquad \theta = \pi.$$

A grid-scale perturbation

$$c_i = 1 + \varepsilon(-1)^i, \qquad \theta = \pi.$$

On an even periodic grid the perturbation sums to zero.

Unstable diffusion steps amplify noise



Diffusive amplification

$$\varepsilon^{n+1} = (1 - 4\mu)\varepsilon^n.$$

Diffusive amplification

$$\varepsilon^{n+1} = (1 - 4\mu)\varepsilon^n.$$

For $\mu = .4$ the factor is $-.6$; for $\mu = .6$ it is -1.4.

Conservation is compatible with instability

The alternating mode remains mean-zero while its magnitude grows.

Conservation is compatible with instability

The alternating mode remains mean-zero while its magnitude grows.

$$|1 - 4\mu| \leq 1 \quad \Longleftrightarrow \quad 0 \leq \mu \leq \tfrac{1}{2}.$$

The scope of Fourier analysis

Constant coefficients and periodic translations make the modes independent.

The scope of Fourier analysis

Constant coefficients and periodic translations make the modes independent. A boundary, variable coefficient or nonlinear reaction introduces additional structure that this calculation does not analyze.