Computational Cardiology Lecture 7

Planar viscous flow and boundary-value discretization

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The incompressible balance

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These equations constrain volume transport, not its magnitude.

A fully developed velocity

$$\boldsymbol{u} = (u(y), 0).$$

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Both divergence and convective acceleration vanish for this ansatz. Momentum must still select u.

Reduced momentum

$$\mu u'' = -G, \qquad G = -\frac{\mathrm{d}p}{\mathrm{d}x}.$$

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Positive G drives positive downstream flow.

Two integrations

$$u(y) = -\frac{G}{2\mu}y^2 + Ay + B.$$

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The two constants are fixed by $u(0) = U_0$ and $u(H) = U_H$.

The Couette–Poiseuille profile

$$u(y) = \frac{G}{2\mu} y(H - y) + U_0 + \frac{U_H - U_0}{H} y.$$

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The pressure-driven quadratic and wall-driven linear profiles superpose.

Flux and maximum speed

For stationary walls,

$$u_{\max} = \frac{GH^2}{8\mu}, \qquad q = \frac{GH^3}{12\mu}, \qquad Q = Wq.$$

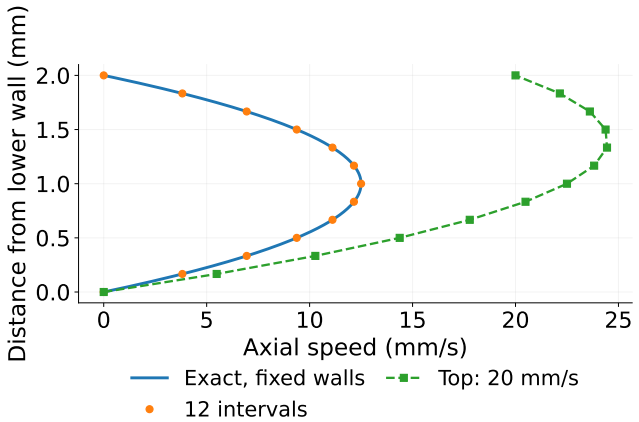
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$$u_{\max} = \frac{GH^2}{8\mu}, \qquad q = \frac{GH^3}{12\mu}, \qquad Q = Wq.$$

q has units m^2/s; Q has units m^3/s.

Boundary values determine the velocity profile



A synthetic channel

$$H = .002 \text{ m}, \quad \mu = .004 \text{ Pa s}, \quad G = 100 \text{ Pa/m}.$$

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$$u_{\max} = .0125 \text{ m/s}, \qquad q = 1.6667 \times 10^{-5} \text{ m}^2/\text{s}.$$

Hydraulic resistance of the planar channel

$$\Delta p = GL, \qquad R = \frac{\Delta p}{Q} = \frac{12\mu L}{WH^3}.$$

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The gap dependence follows from this geometry.

Centered interior equations

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Known wall values move to the right-hand side.

The Dirichlet matrix is positive definite

$$z^{\mathsf{T}} A z = \sum_{j=0}^{N-1} (z_{j+1} - z_j)^2.$$

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With zero endpoints, equality requires $z = 0$.

Exact differentiation of a quadratic

$$\frac{u(y-h)-2u(y)+u(y+h)}{h^2} = u''(y)$$

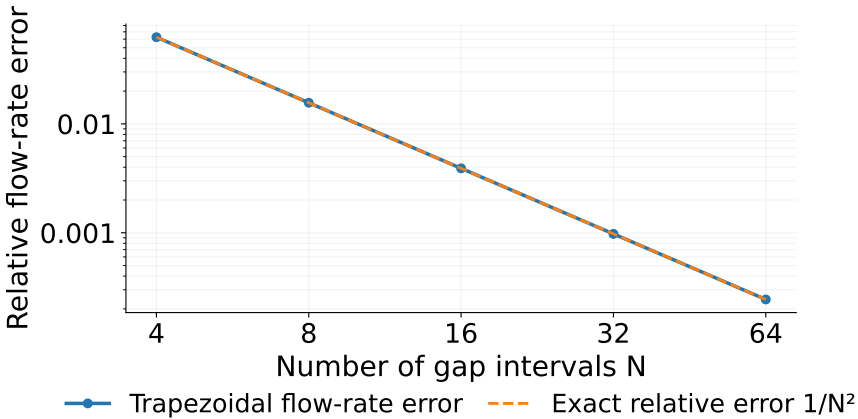
for quadratic u.

Exact differentiation of a quadratic

$$\frac{u(y-h) - 2u(y) + u(y+h)}{h^2} = u''(y)$$

for quadratic u. This benchmark can have roundoff-level nodal error at every grid size.

An exact nodal profile can have an inexact integral



The integral is a separate operation

$$q_h = h \left[\frac{u_0 + u_N}{2} + \sum_{j=1}^{N-1} u_j \right].$$

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Straight segments under a concave parabola underestimate its area.

An exact quadrature-error formula

$$q_h = q(1 - N^{-2}), \qquad \frac{q - q_h}{q} = N^{-2}.$$

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The observed order two belongs to quadrature, not a measured nodal-solve error.

Changing the wall velocity

$$q = \frac{GH^3}{12\mu} + \frac{H}{2}(U_0 + U_H).$$

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Trapezoidal quadrature integrates the linear contribution exactly.

The limits of the reduction

The calculation assumes a long straight channel, fixed viscosity and fully developed flow.

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The calculation assumes a long straight channel, fixed viscosity and fully developed flow. It does not solve an entrance region, moving curved boundaries or general pressure–velocity coupling.