Computational Cardiology Lecture 8

# Streamfunctions on a mapped chamber domain

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# Incompressibility as a kinematic constraint

$$\nabla \cdot \boldsymbol{u} = 0.$$

A streamfunction enforces this constraint without supplying a momentum equation or pressure field.

# The mapped chamber

$$H(x) = H_0[1 + b\sin^2(\pi x/L)], \qquad y = \eta H(x).$$

$0 \le \eta \le 1$ maps a rectangle to a smooth widening domain.

# Integrate the volume

$$V = W \int_0^L H(x)\,\mathrm{d}x = W H_0 L(1 + b/2).$$

The supplied dimensions give $V = 21$ mL.

# A through-flow streamfunction

$$\psi_0 = qf(\eta), \qquad f(\eta) = 3\eta^2 - 2\eta^3.$$

$$f(1) - f(0) = 1, \qquad f'(0) = f'(1) = 0.$$

# An interior perturbation

$$\psi = \psi_0 + \alpha q \sin^2(\pi x/L)\eta^2(1-\eta)^2.$$

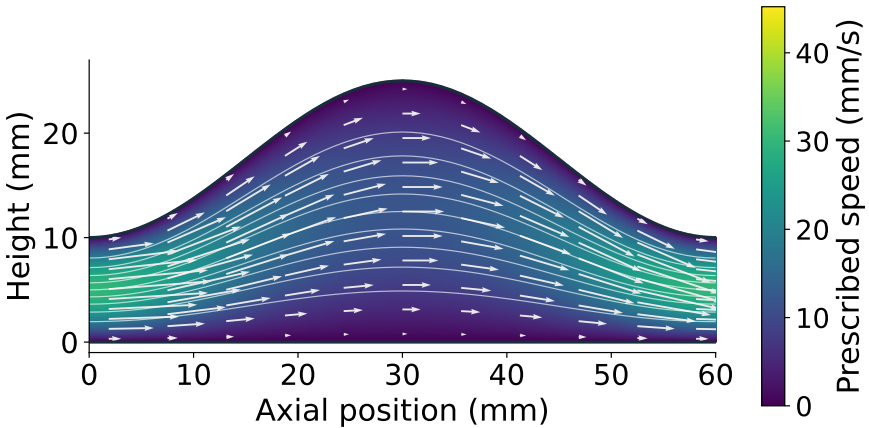
The perturbation preserves the port flow and vanishes with the required wall derivatives.

# Differentiate at the correct fixed coordinate

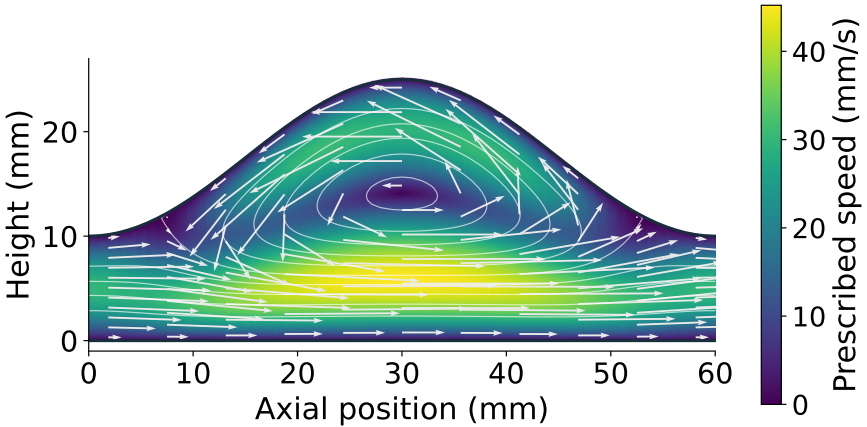
$$\partial_y = H^{-1}\partial_\eta, \qquad \partial_x|_y = \partial_x|_\eta - \frac{\eta H'}{H}\partial_\eta.$$

The second term accounts for the changing height.

# A: through-flow $\alpha = 0$



# B: circulation $\alpha = 24$



# The physical velocity

$$u = \frac{\psi_\eta}{H}, \qquad v = -\psi_x + \frac{\eta H'}{H}\psi_\eta.$$

Omitting the map term changes the field.

# Cancellation of mixed derivatives

$$\partial_x u + \partial_y v = \partial_{xy}\psi - \partial_{yx}\psi = 0.$$

Smoothness gives incompressibility. It does not give momentum balance.

# Flux through a cross-section

$$q = \psi(x, H(x)) - \psi(x, 0), \qquad Q = Wq.$$

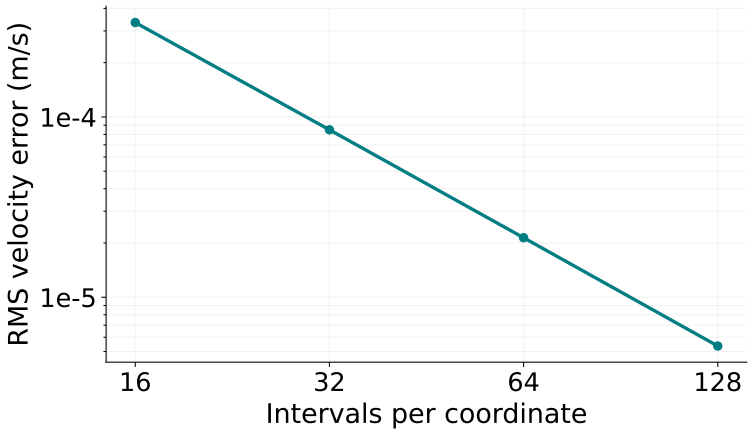
Interior changes can leave this difference unchanged.

# A closed-cell flux identity

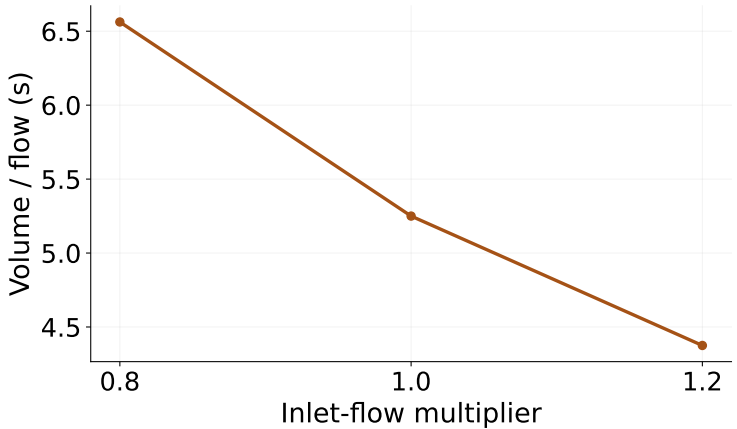
$$\oint_{\partial K} \boldsymbol{u} \cdot n \, \mathrm{d}s = \oint_{\partial K} \mathrm{d}\psi = 0.$$

Corner values telescope for consistently oriented edge fluxes.

# Derivative refinement



# Input sensitivity



# Sampled derivatives form a different calculation

$$\nabla \cdot u = u_x|_\eta - \frac{\eta H'}{H} u_\eta + \frac{1}{H} v_\eta.$$

Derivative reconstruction has a mesh error even when corner fluxes cancel exactly.

# The nullspace of a port-flow observation

$$\psi_\alpha = \psi_0 + \alpha\chi, \qquad \mathcal{Q}(\chi) = 0.$$

Then $\mathcal{Q}(\psi_\alpha) = q$ for every $\alpha$, although the interior velocities differ.
Here $\mathcal{Q}(\psi) = \psi(x, H(x)) - \psi(x, 0)$ is flux per unit width; total flow is $Q = W\,\mathcal{Q}(\psi)$.

# A bulk turnover scale

$$T_{\text{bulk}} = \frac{V}{Q} = \frac{21\ \text{mL}}{4\ \text{mL/s}} = 5.25\ \text{s}.$$

This ratio does not assign a residence time to each particle.

# Boundary-input sensitivity

$$T(0.8Q) = 6.5625 \text{ s}, \qquad T(1.2Q) = 4.375 \text{ s}.$$

Refining numerical derivatives cannot remove uncertainty in the prescribed input.

# The missing dynamical constraint

A viscous-flow calculation also requires pressure, acceleration, stress and compatible forcing or boundary data. The streamfunction fixture supplies exact kinematics, not a solved atrial CFD model.