Computational Cardiology Lecture 8

Streamfunctions on a mapped chamber domain

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Incompressibility as a kinematic constraint

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A streamfunction enforces this constraint without supplying a momentum equation or pressure field.

The mapped chamber

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$0 \le \eta \le 1$ maps a rectangle to a smooth widening domain.

Integrate the volume

$$V = W \int_0^L H(x)\,\mathrm{d}x = W H_0 L(1 + b/2).$$

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The supplied dimensions give $V = 21$ mL.

A through-flow streamfunction

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$$f(1) - f(0) = 1, \qquad f'(0) = f'(1) = 0.$$

An interior perturbation

$$\psi = \psi_0 + \alpha q \sin^2(\pi x/L)\eta^2(1-\eta)^2.$$

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The perturbation preserves the port flow and vanishes with the required wall derivatives.

Differentiate at the correct fixed coordinate

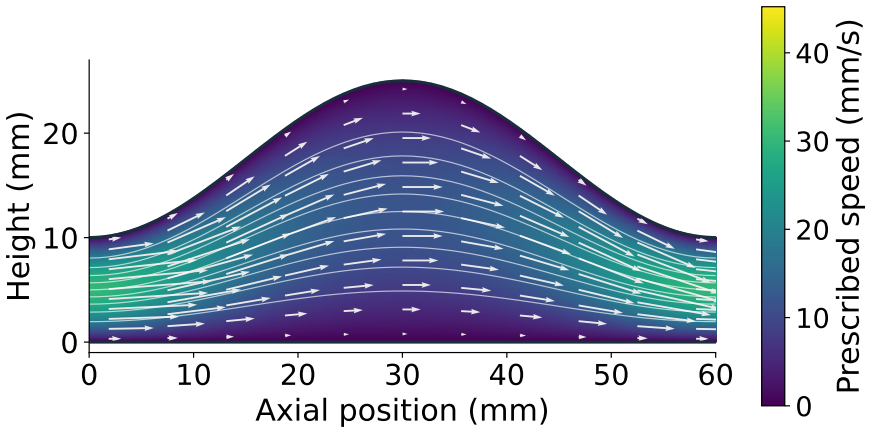
$$\partial_y = H^{-1}\partial_\eta, \qquad \partial_x|_y = \partial_x|_\eta - \frac{\eta H'}{H}\partial_\eta.$$

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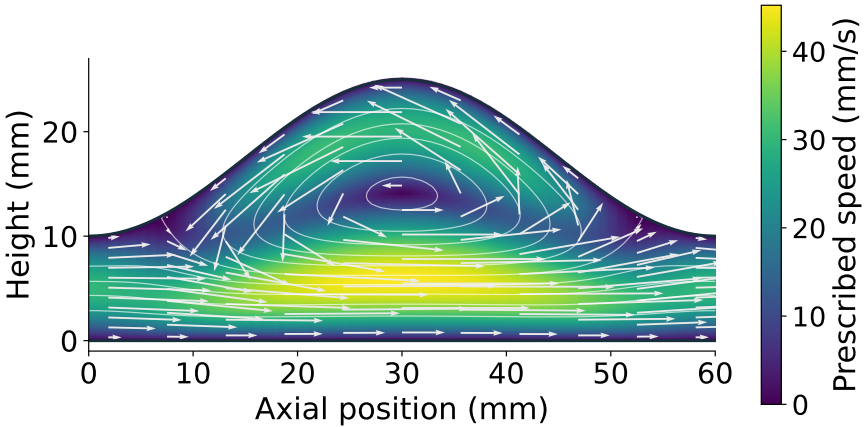
$$\partial_y = H^{-1}\partial_\eta, \qquad \partial_x|_y = \partial_x|_\eta - \frac{\eta H'}{H}\partial_\eta.$$

The second term accounts for the changing height.

A: through-flow $\alpha = 0$



B: circulation $\alpha = 24$



The physical velocity

$$u = \frac{\psi_\eta}{H}, \qquad v = -\psi_x + \frac{\eta H'}{H}\psi_\eta.$$

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Omitting the map term changes the field.

Cancellation of mixed derivatives

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Smoothness gives incompressibility. It does not give momentum balance.

Flux through a cross-section

$$q = \psi(x, H(x)) - \psi(x, 0), \qquad Q = Wq.$$

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Interior changes can leave this difference unchanged.

A closed-cell flux identity

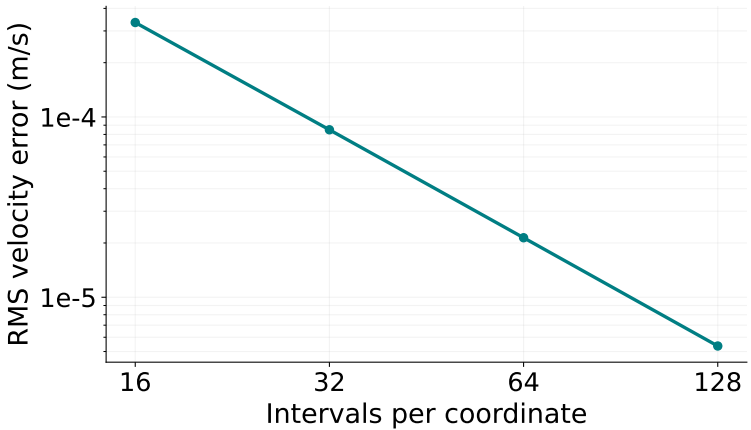
$$\oint_{\partial K} \boldsymbol{u} \cdot n \, \mathrm{d}s = \oint_{\partial K} \mathrm{d}\psi = 0.$$

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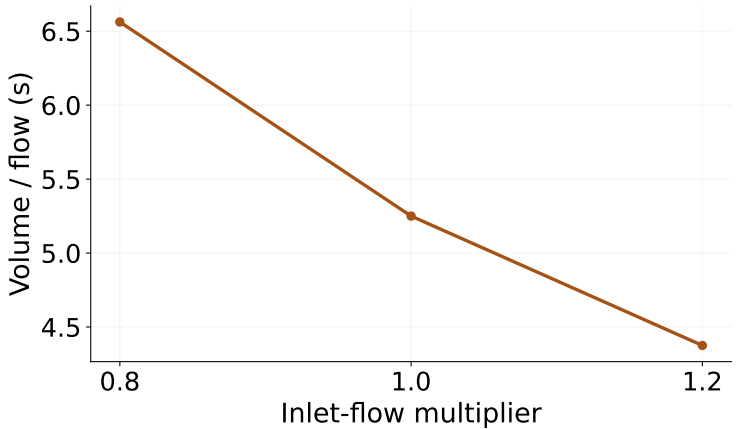
$$\oint_{\partial K} \boldsymbol{u} \cdot n \, \mathrm{d}s = \oint_{\partial K} \mathrm{d}\psi = 0.$$

Corner values telescope for consistently oriented edge fluxes.

Derivative refinement



Input sensitivity



Sampled derivatives form a different calculation

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Derivative reconstruction has a mesh error even when corner fluxes cancel exactly.

The nullspace of a port-flow observation

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Then $\mathcal{Q}(\psi_\alpha) = q$ for every α, although the interior velocities differ.
Here $\mathcal{Q}(\psi) = \psi(x, H(x)) - \psi(x, 0)$ is flux per unit width; total flow is $Q = W\,\mathcal{Q}(\psi)$.

A bulk turnover scale

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This ratio does not assign a residence time to each particle.

Boundary-input sensitivity

$$T(0.8Q) = 6.5625 \text{ s}, \qquad T(1.2Q) = 4.375 \text{ s}.$$

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Refining numerical derivatives cannot remove uncertainty in the prescribed input.

The missing dynamical constraint

A viscous-flow calculation also requires pressure, acceleration, stress and compatible forcing or boundary data.

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A viscous-flow calculation also requires pressure, acceleration, stress and compatible forcing or boundary data. The streamfunction fixture supplies exact kinematics, not a solved atrial CFD model.