Computational Cardiology Lecture 9

Membrane current balance and voltage-dependent gates

Zan Ahmad

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Membrane charge balance

$$C\dot{V} = I_{\text{stim}} - \sum_j g_j(V - E_j).$$

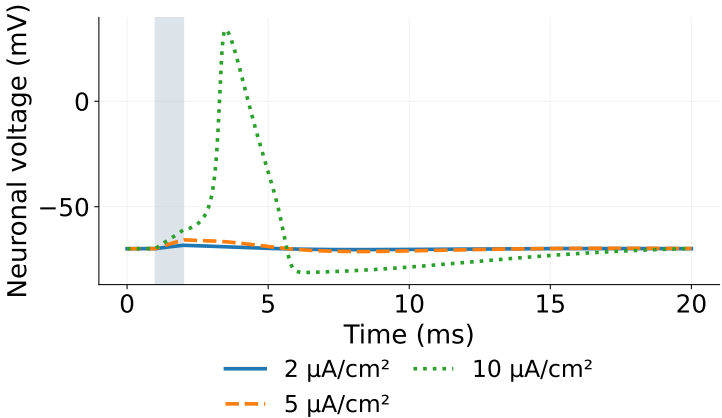
Outward ionic current is positive; applied inward current is positive.

Densities and total current

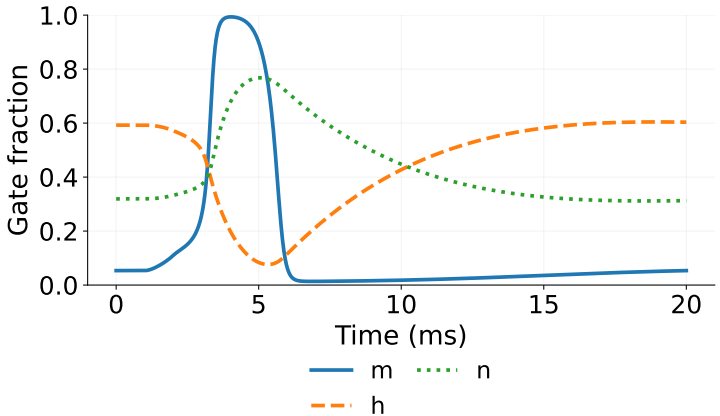
$$I_{\text{density}} = I_{\text{total}}/A.$$

$$(\mu\text{F/cm}^2)(\text{mV/ms}) = \mu\text{A/cm}^2.$$

Equal-duration neuronal pulses



Bounded gate fractions



Voltage driving forces and conductances

$$g_{Na} = 120m^3h, \qquad g_K = 36n^4, \qquad g_L = .3.$$

$$(E_{Na}, E_K, E_L) = (45, -82, -59) \text{ mV}.$$

The gate equation

$$\dot{s} = \alpha_s(V)(1 - s) - \beta_s(V)s.$$

$$s_\infty = \frac{\alpha_s}{\alpha_s + \beta_s}, \qquad \tau_s = \frac{1}{\alpha_s + \beta_s}.$$

Bounds of the continuous gate

At $s = 0$, $\dot{s} = \alpha_s \geq 0$. At $s = 1$, $\dot{s} = -\beta_s \leq 0$. The vector field points into the admissible interval.

The removable rate singularity

$$f(z) = \frac{z}{1 - e^{-z}}, \qquad f(0) = 1.$$

$$f(z) = 1 + z/2 + z^2/12 - z^4/720 + O(z^6).$$

Two activation rates

$$\alpha_m = f((V+45)/10), \qquad \alpha_n = .1f((V+60)/10).$$

At $V = -45$ and -60 mV, the respective rates are 1 and $.1\ \mathrm{ms}^{-1}$.

An exact frozen-voltage gate step

For $k_s = \alpha_s + \beta_s > 0$,

$$s^{n+1} = s_\infty + (s^n - s_\infty)e^{-k_s \Delta t}.$$

This is a convex combination of s^n and s_∞ for nonnegative rates.

The zero-rate limit

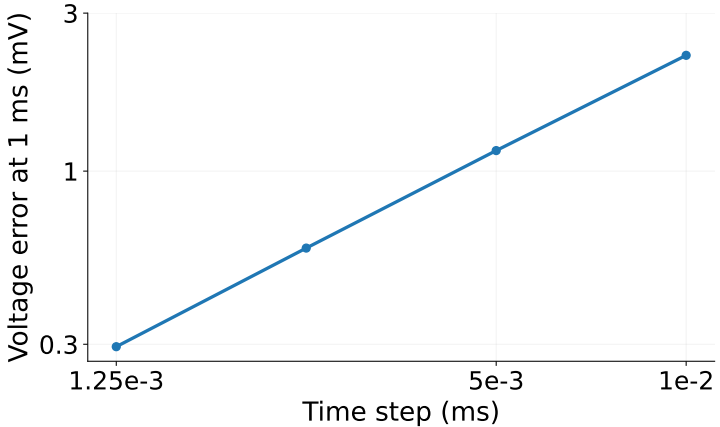
If $\alpha_s = \beta_s = 0$, then $\dot{s} = 0$ and $s^{n+1} = s^n$. No equilibrium fraction needs to be formed by division by zero.

Freeze conductances after the gate step

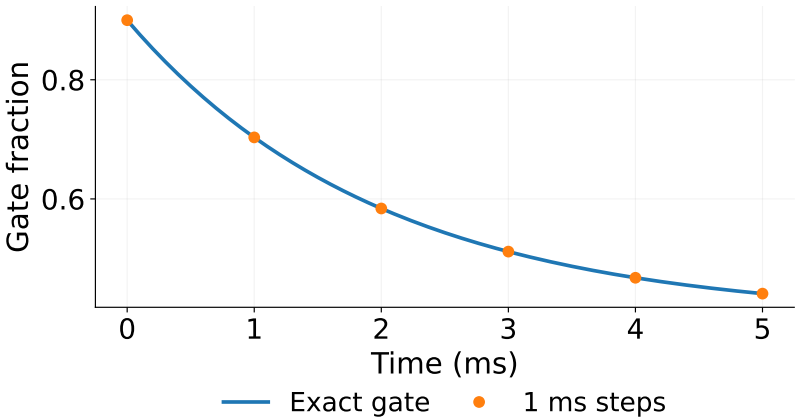
$$G = \sum_j g_j, \qquad B = \sum_j g_j E_j + I.$$

$$C\dot{V} = B - GV, \qquad V_\infty = B/G.$$

First-order coupled error



Frozen-rate gate check



An exact frozen-conductance voltage step

$$V^{n+1} = V_\infty + (V^n - V_\infty)e^{-G\Delta t/C}.$$

The two exact substeps form a first-order coupled method.

The coupled rest equation

$$\mathcal{I}(V) = \sum_j g_j(s_\infty(V))(V - E_j) = 0.$$

Resting gates and zero net current must hold at the same voltage.

Current cancellation at rest

Individual ionic currents can remain nonzero while their sum vanishes. The companion recomputes the equilibrium from its exact rate constants, near -69.89767 mV.

Excitation depends on the protocol

Pulse amplitude, duration, initial voltage and initial gates jointly determine the response. A zero-crossing distinction for one pulse does not define a universal excitation threshold.

Model identity

These neuronal kinetics describe a specific V, m, h, n system. A reduced cardiac model changes both the variables and the equations; it is not obtained by relabeling a neuronal trace.