Computational Cardiology Lecture 10

A reduced cardiac reaction–diffusion wave

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A scaled cardiac reaction system

$$v_t = \frac{hv^2(1-v)}{\tau_{in}} - \frac{v}{\tau_{out}} + I + Dv_{xx}.$$

v, h are dimensionless; time is ms and distance is cm.

The recovery gate

$$h_t = \begin{cases} (1-h)/\tau_{open}, & v < v_{gate}, \\ -h/\tau_{close}, & v \geq v_{gate}. \end{cases}$$

Equality uses the closing branch.

The resting state

$$(v, h) = (0, 1), \qquad I = 0.$$

Both local derivatives vanish. Homogeneous diffusion also vanishes.

Shared diffusive face fluxes

$$F_{i+1/2} = -D\frac{v_{i+1} - v_i}{\Delta x}, \qquad \dot{v}_i = \frac{F_{i-1/2} - F_{i+1/2}}{\Delta x}.$$

Sealed endpoints have zero face flux.

The chain Laplacian

$$\dot{v} = -L_h v, \qquad L_h \mathbf{1} = 0.$$

$$v^{\mathsf{T}} L_h v = \frac{D}{\Delta x^2} \sum_i (v_{i+1} - v_i)^2 \geq 0.$$

Exact frozen-branch gate evolution

Below threshold, $h^* = 1 + (h^n - 1)e^{-\Delta t/\tau_{open}}$. Above threshold, $h^* = h^n e^{-\Delta t/\tau_{close}}$. Both preserve $[0, 1]$.

A semi-implicit reaction step

$$a = \frac{h^*(v^n)^2}{\tau_{in}}, \qquad v^* = \frac{v^n + \Delta t(a + I)}{1 + \Delta t(a + 1/\tau_{out})}.$$

Activation is frozen while the remaining linear factors are implicit.

Prove the upper bound

$$\text{denominator} - \text{numerator} = 1 - v^n + \Delta t(1/\tau_{out} - I).$$

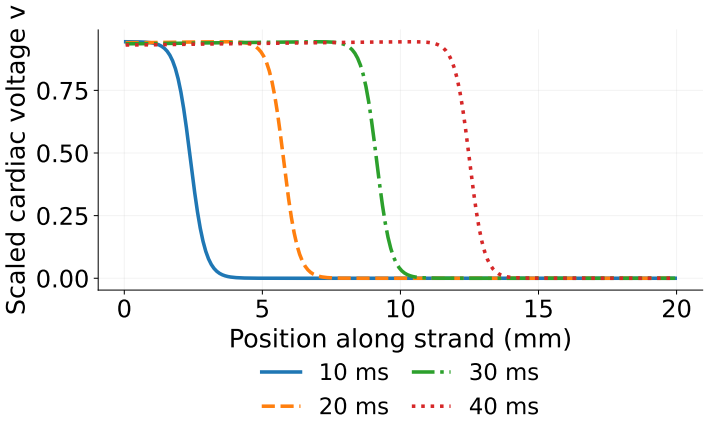
It is nonnegative if $v^n \leq 1$ and $I \leq 1/\tau_{out}$.

Implicit diffusion preserves the range

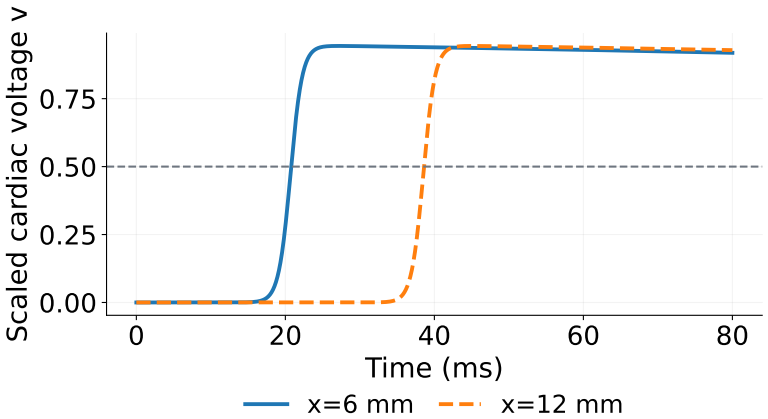
$$(I + \Delta t L_h) v^{n+1} = v^*.$$

The inverse is nonnegative and preserves constants, so it maps bounded data into the same interval.

Reduced cardiac front



Two physical sensors



A fixed spatial source

$$I_i = I_0 \frac{|K_i \cap [0, \ell_s]|}{\Delta x}.$$

Fractional overlap preserves the source width under mesh refinement.

A fixed temporal source

$$\int\int I \, \mathrm{d}x \, \mathrm{d}t = I_0 \ell_s T_s = .02 \text{ cm}.$$

These are scaled-voltage integral units, not microcoulombs.

Interpolate an upward crossing

$$t_* = t_0 + \frac{v_* - v_0}{v_1 - v_0}(t_1 - t_0).$$

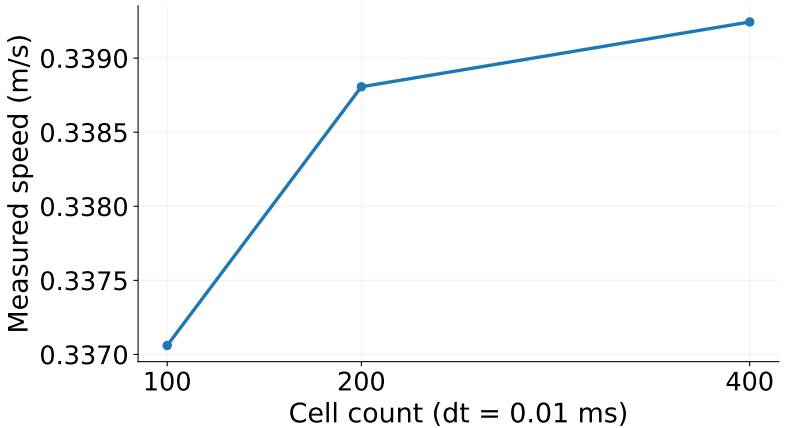
A valid upward bracket and unequal endpoint values are required.

Define the measured speed

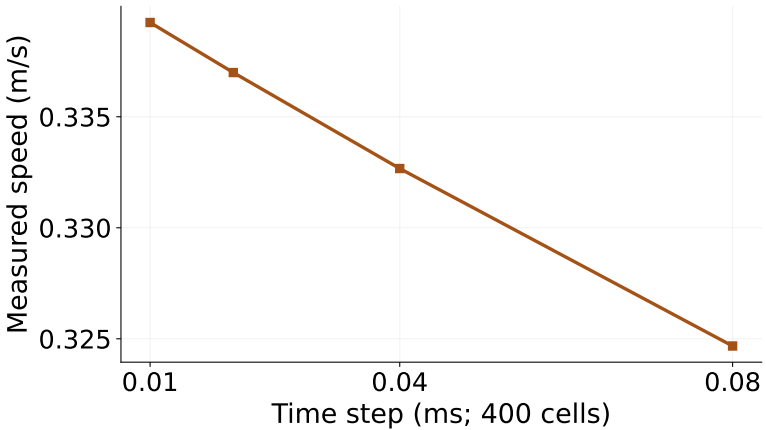
$$c = \frac{x_2 - x_1}{t_{*,2} - t_{*,1}}.$$

Sensors at $.6$ and 1.2 cm use the same $v_* = .5$ criterion.

Refine space



Refine time


Measured speed (m/s)
0.335
0.330
0.325
0.01
0.04
0.08
Time step (ms; 400 cells)

Timing error propagates into speed

$$\frac{\delta c}{c} \simeq -\frac{\delta \Delta t_s}{\Delta t_s}$$

for fixed sensor separation. Spatial and temporal refinement must retain the same stimulus and sensor definitions.

Scope of the propagation result

The homogeneous sealed strand omits anisotropy, branching and extracellular potential. Bounded states and a plausible speed do not establish a tissue-specific physiological calibration.