Computational Cardiology Lecture 10

A reduced cardiac reaction–diffusion wave

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A scaled cardiac reaction system

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v, h are dimensionless; time is ms and distance is cm.

The recovery gate

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Equality uses the closing branch.

The resting state

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Both local derivatives vanish. Homogeneous diffusion also vanishes.

Shared diffusive face fluxes

$$F_{i+1/2} = -D\frac{v_{i+1} - v_i}{\Delta x}, \qquad \dot{v}_i = \frac{F_{i-1/2} - F_{i+1/2}}{\Delta x}.$$

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Sealed endpoints have zero face flux.

The chain Laplacian

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$$v^{\mathsf{T}} L_h v = \frac{D}{\Delta x^2} \sum_i (v_{i+1} - v_i)^2 \geq 0.$$

Exact frozen-branch gate evolution

Below threshold, $h^* = 1 + (h^n - 1)e^{-\Delta t/\tau_{open}}$.

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Below threshold, $h^* = 1 + (h^n - 1)e^{-\Delta t/\tau_{open}}$. Above threshold, $h^* = h^n e^{-\Delta t/\tau_{close}}$. Both preserve $[0, 1]$.

A semi-implicit reaction step

$$a = \frac{h^*(v^n)^2}{\tau_{in}}, \qquad v^* = \frac{v^n + \Delta t(a + I)}{1 + \Delta t(a + 1/\tau_{out})}.$$

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Activation is frozen while the remaining linear factors are implicit.

Prove the upper bound

$$\text{denominator} - \text{numerator} = 1 - v^n + \Delta t(1/\tau_{out} - I).$$

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It is nonnegative if $v^n \leq 1$ and $I \leq 1/\tau_{out}$.

Implicit diffusion preserves the range

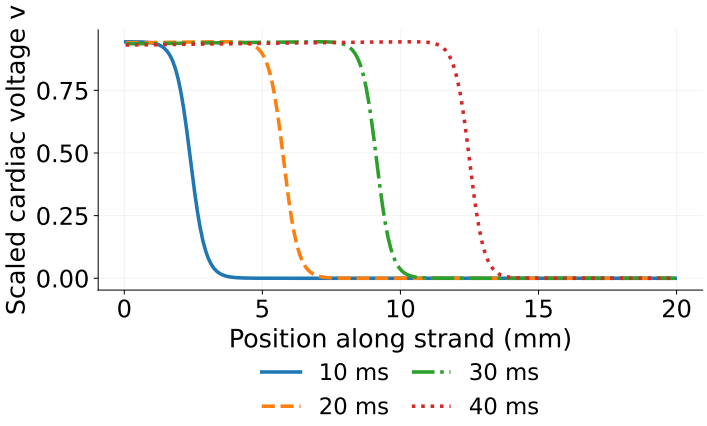
$$(I + \Delta t L_h) v^{n+1} = v^*.$$

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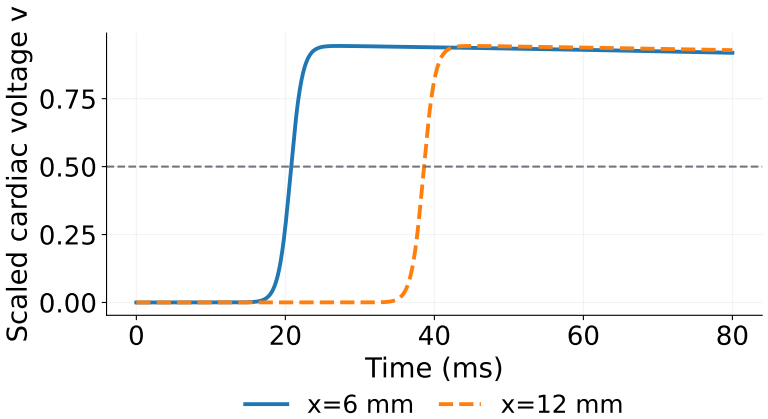
$$(I + \Delta t L_h) v^{n+1} = v^*.$$

The inverse is nonnegative and preserves constants, so it maps bounded data into the same interval.

Reduced cardiac front



Two physical sensors



A fixed spatial source

$$I_i = I_0 \frac{|K_i \cap [0, \ell_s]|}{\Delta x}.$$

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Fractional overlap preserves the source width under mesh refinement.

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These are scaled-voltage integral units, not microcoulombs.

Interpolate an upward crossing

$$t_* = t_0 + \frac{v_* - v_0}{v_1 - v_0}(t_1 - t_0).$$

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A valid upward bracket and unequal endpoint values are required.

Define the measured speed

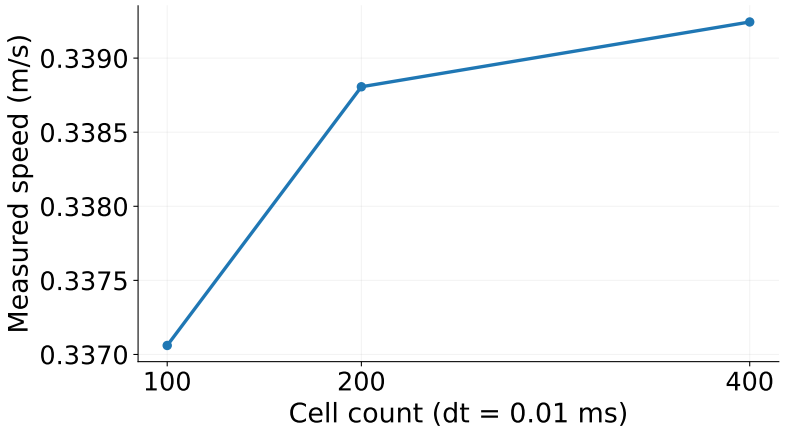
$$c = \frac{x_2 - x_1}{t_{*,2} - t_{*,1}}.$$

Define the measured speed

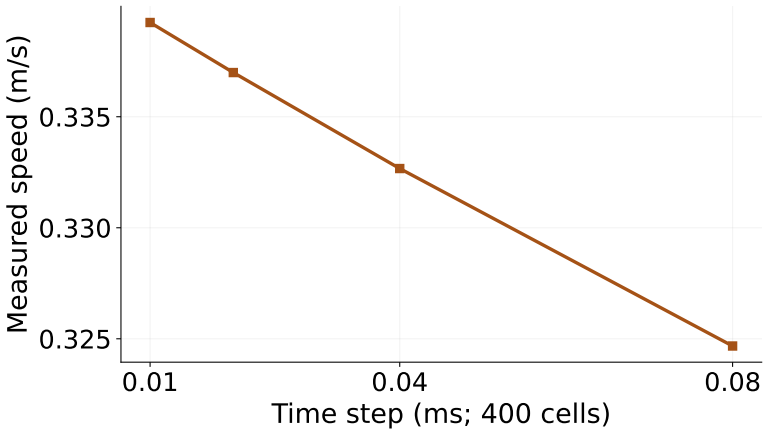
$$c = \frac{x_2 - x_1}{t_{*,2} - t_{*,1}}.$$

Sensors at $.6$ and 1.2 cm use the same $v_* = .5$ criterion.

Refine space



Refine time


Measured speed (m/s)
0.335
0.330
0.325
0.01
0.04
0.08
Time step (ms; 400 cells)

Timing error propagates into speed

$$\frac{\delta c}{c} \simeq -\frac{\delta \Delta t_s}{\Delta t_s}$$

for fixed sensor separation.

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for fixed sensor separation. Spatial and temporal refinement must retain the same stimulus and sensor definitions.

Scope of the propagation result

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The homogeneous sealed strand omits anisotropy, branching and extracellular potential. Bounded states and a plausible speed do not establish a tissue-specific physiological calibration.