Computational Cardiology Lecture 13

Rigid alignment and arclength representations of closed curves

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A cyclic polygon in physical coordinates

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The final edge joins x_{n-1} to x_0 without duplicating the first row.

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Division by n does not repair an incorrect correspondence.

The transformations removed

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Physical scale and reflection are retained.

Eliminate the translation

$$X_c = X - \mathbf{1}\bar{x}, \quad Y_c = Y - \mathbf{1}\bar{y}, \quad t = \bar{x} - \bar{y}R.$$

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The vertex centroid depends on sampling density.

Expand the least-squares objective

$$\|X_c - Y_c R\|_F^2 = \|X_c\|_F^2 + \|Y_c\|_F^2 - 2\,\mathrm{tr}(R^{\mathsf{T}} Y_c^{\mathsf{T}} X_c).$$

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Only the final term depends on R.

The cross-product matrix

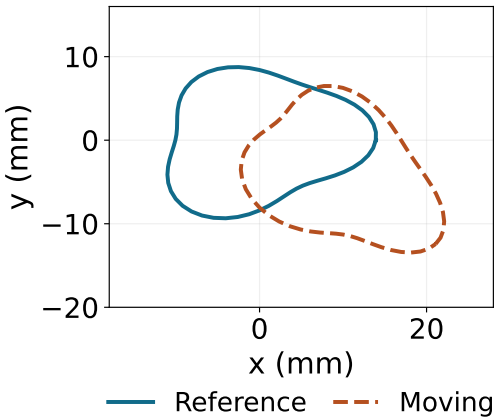
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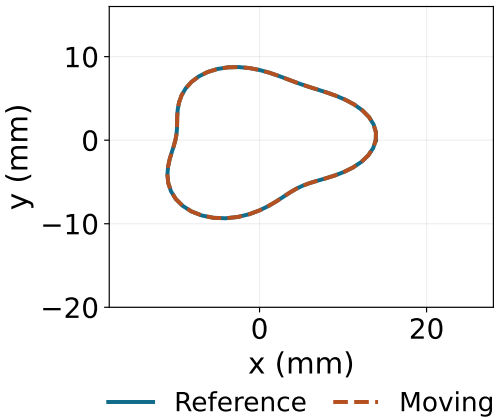
$$Y_c^{\mathsf{T}} X_c = U \Sigma V^{\mathsf{T}}.$$

The singular vectors identify the best paired orientation in this two-dimensional problem.

Before alignment



Proper rigid alignment



Enforce a proper rotation

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The determinant correction excludes a mirror transformation.

Retaining physical size

For centered, consistently oriented points,

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Rigid alignment does not normalize scale.

Point density changes index correspondence

Equal-length arrays can sample different positions along the same curve.

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Equal-length arrays can sample different positions along the same curve. A vertex index is not automatically a common anatomical location.

Cumulative polygonal arclength

$$s_k = \sum_{j<k} \|p_{j+1} - p_j\|, \qquad L = \sum_{j=0}^{n-1} \|p_{j+1} - p_j\|.$$

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The sums include the closing edge.

Equal-distance targets

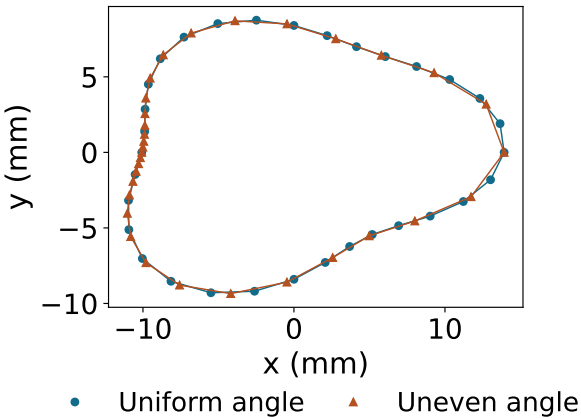
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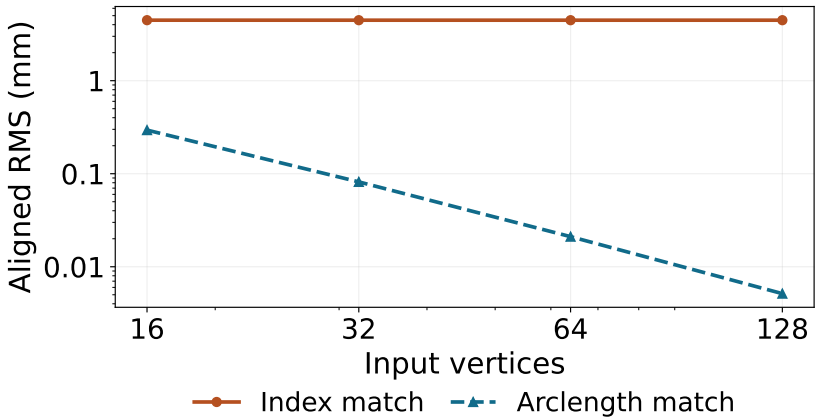
$$q_\ell = \ell L/n, \qquad \ell = 0, \ldots, n-1.$$

The target L is excluded because it duplicates the start.

Same analytic contour



Sampling artifact



Interpolate within an edge

$$\alpha = \frac{q - s_k}{s_{k+1} - s_k}, \qquad x(q) = (1 - \alpha)p_k + \alpha p_{k+1}.$$

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Interpolation cannot recover an unresolved bend.

Zero-length edges and zero spread

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Nonunique alignment

A degenerate cross-product matrix can make the minimizing rotation nonunique.

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A degenerate cross-product matrix can make the minimizing rotation nonunique. A minimizing transformation describes the stated paired coordinates; it does not establish anatomical correspondence.

Residual components

After pose removal, a discrepancy may reflect size, local deformation, sampling or segmentation.

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After pose removal, a discrepancy may reflect size, local deformation, sampling or segmentation. Arclength fractions supply boundary positions, not validated landmarks.