Computational Cardiology Lecture 14

Shape comparisons, deformation penalties and geometric matching

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A comparison requires an invariance choice

Raw coordinate RMS includes pose, cyclic start and shape. We remove proper rigid motion and cyclic start while retaining physical scale and traversal direction.

A known finite cohort

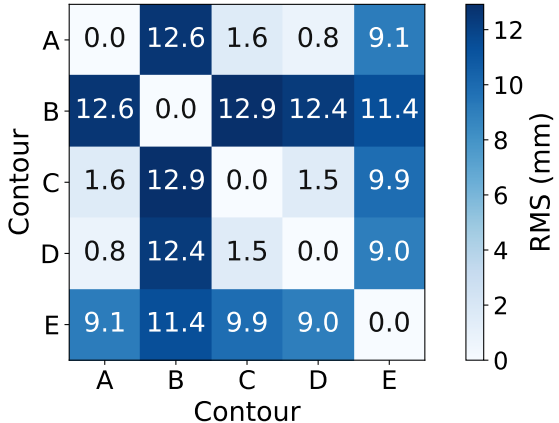
A is the reference; B is a rigid copy; C is 15% larger; D has a local displacement; E has a shifted starting index. These relationships are constructed, not discovered.

Cyclic rigid discrepancy

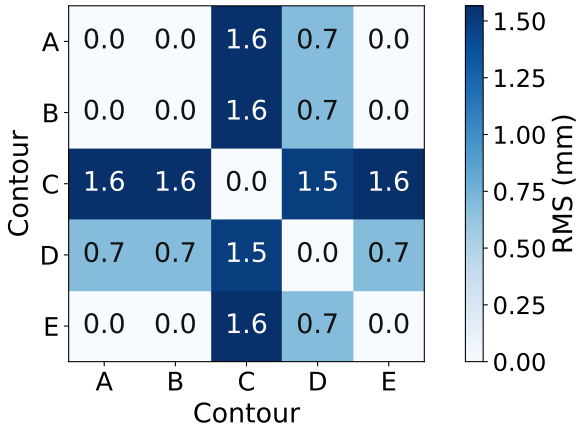
$$d_{rc}(X,Y) = \min_{k,R,t} \left(\frac{1}{n} \sum_{j} \|x_j - (y_{j+k}R + t)\|^2 \right)^{1/2}.$$

$R \in SO(2)$ and indices wrap modulo n.

Raw index RMS



Rigid + cyclic RMS



Enumerate the starting index

For each k, center the arrays, solve the proper Procrustes problem, and retain the smallest residual. The identity choice guarantees $d_{rc} \leq d_{\mathrm{RMS}}$.

Retained geometric differences

Rigid copies and cyclic reindexings can have zero aligned discrepancy. Size and local displacement remain nonzero because neither is removed by the allowed transformations.

Equal RMS does not imply equal displacement pattern

$$\frac{1}{n}\sum_{j}\|u_j\|^2$$

measures amplitude but ignores how neighboring vectors vary. Smooth and alternating fields can have the same value.

A quadratic neighbor penalty

$$E_\lambda(u) = \frac{1}{n}\sum_j \|u_j\|^2 + \frac{\lambda}{n}\sum_j \|u_{j+1} - u_j\|^2.$$

E_λ has units mm^2; $\lambda \geq 0$ is dimensionless.

Matrix form of the score

$$E_\lambda(u) = \frac{1}{n} u^\mathsf{T} (I + \lambda D^\mathsf{T} D) u.$$

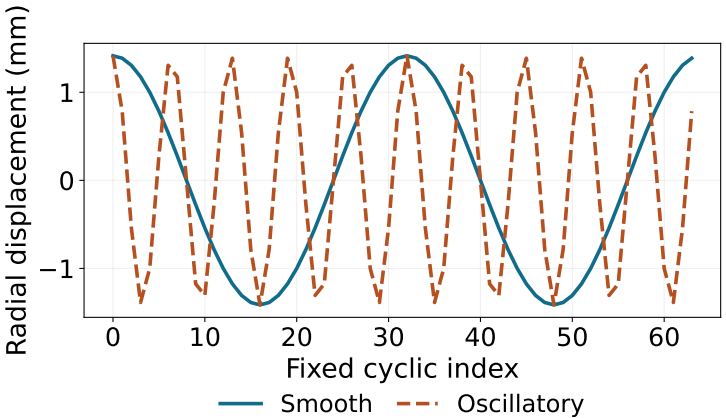
The identity term makes the matrix positive definite.

Scaling and constant shifts

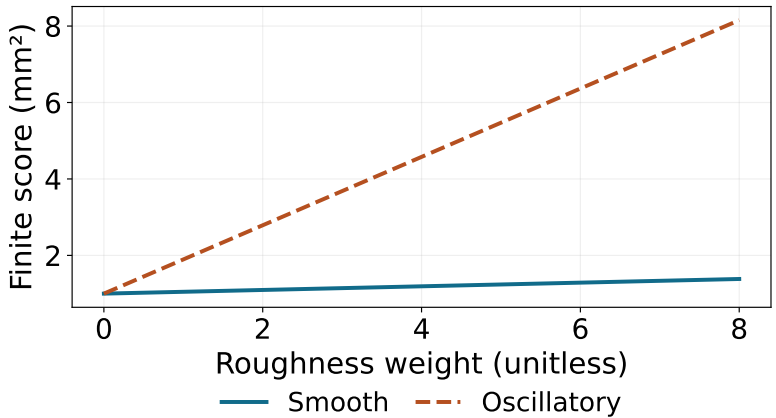
$$E_\lambda(au) = a^2 E_\lambda(u).$$

A constant displacement has zero neighbor penalty but nonzero amplitude cost.

Both RMS = 1 mm


Radial displacement (mm)
1
0
−1
0
10
20
30
40
50
60
Fixed cyclic index
Smooth
Oscillatory

Same size, different roughness



Fourier weights of roughness

$$E_\lambda = \sum_k [1 + 4\lambda \sin^2(\pi k/n)] \|\widehat{u}_k\|^2.$$

Higher-frequency displacement patterns receive greater penalty.

The grid is part of this definition

The difference $u_{j+1} - u_j$ is not divided by arclength. Changing sample density changes its interpretation as a derivative penalty.

Elastic energy along a path

$$\mathcal{E}(c) = \int_0^1 G_{c(t)}(\dot{c}(t), \dot{c}(t))\,\mathrm{d}t.$$

A path metric measures deformation throughout a trajectory in shape space.

Endpoint mismatch is a separate term

$$\mathcal{E}(c) + \gamma \|\mu_{c(1)} - \mu_{target}\|_K^2.$$

The endpoint penalty promotes matching; it is not interchangeable with the path energy.

Nonoriented geometric comparison

Positions, geometric weights and tangent or normal directions define an endpoint representation. Squared directional dot products can identify opposite orientations. Kernel scale controls spatial sensitivity.

The implemented and contextual calculations

The companion computes finite cyclic alignment and a quadratic displacement score. Elastic surface paths and varifold registration remain research context, with additional geometry, optimization and numerical requirements.