Computational Cardiology Lecture 14

Shape comparisons, deformation penalties and geometric matching

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A comparison requires an invariance choice

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Raw coordinate RMS includes pose, cyclic start and shape. We remove proper rigid motion and cyclic start while retaining physical scale and traversal direction.

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A is the reference; B is a rigid copy; C is 15% larger; D has a local displacement; E has a shifted starting index.

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A is the reference; B is a rigid copy; C is 15% larger; D has a local displacement; E has a shifted starting index. These relationships are constructed, not discovered.

Cyclic rigid discrepancy

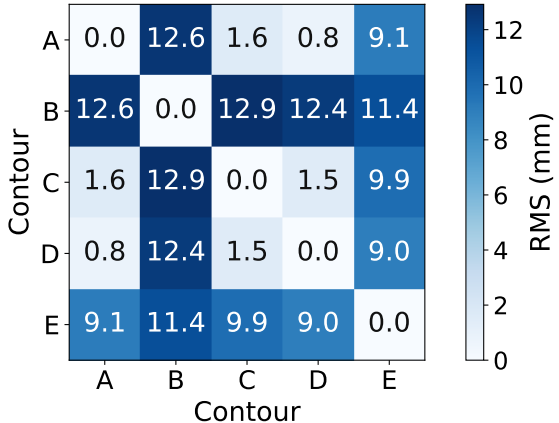
$$d_{rc}(X,Y) = \min_{k,R,t} \left(\frac{1}{n} \sum_{j} \|x_j - (y_{j+k}R + t)\|^2 \right)^{1/2}.$$

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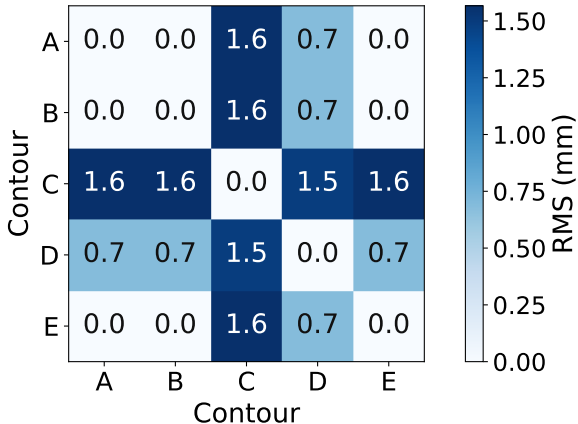
$$d_{rc}(X,Y) = \min_{k,R,t} \left(\frac{1}{n} \sum_{j} \|x_j - (y_{j+k}R + t)\|^2 \right)^{1/2}.$$

$R \in SO(2)$ and indices wrap modulo n.

Raw index RMS



Rigid + cyclic RMS



Enumerate the starting index

For each k, center the arrays, solve the proper Procrustes problem, and retain the smallest residual.

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Retained geometric differences

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Rigid copies and cyclic reindexings can have zero aligned discrepancy. Size and local displacement remain nonzero because neither is removed by the allowed transformations.

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measures amplitude but ignores how neighboring vectors vary.

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measures amplitude but ignores how neighboring vectors vary. Smooth and alternating fields can have the same value.

A quadratic neighbor penalty

$$E_\lambda(u) = \frac{1}{n}\sum_j \|u_j\|^2 + \frac{\lambda}{n}\sum_j \|u_{j+1} - u_j\|^2.$$

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E_λ has units mm^2; $\lambda \geq 0$ is dimensionless.

Matrix form of the score

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The identity term makes the matrix positive definite.

Scaling and constant shifts

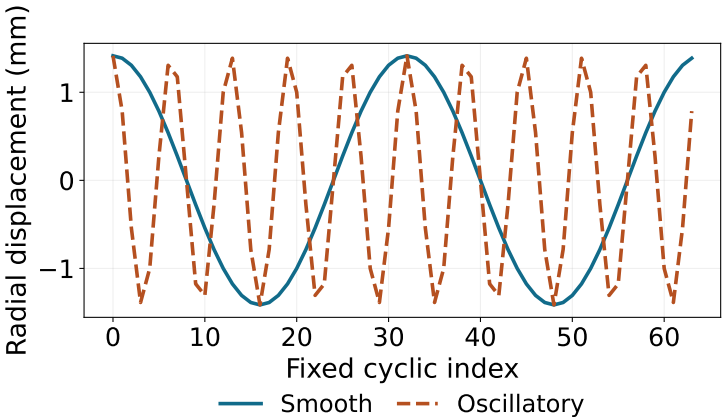
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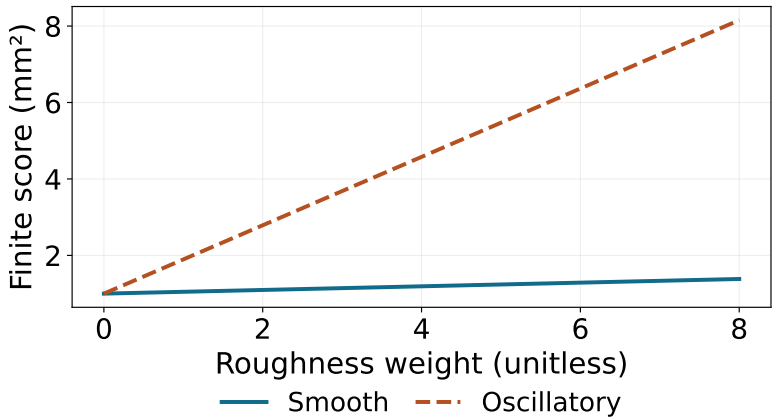
$$E_\lambda(au) = a^2 E_\lambda(u).$$

A constant displacement has zero neighbor penalty but nonzero amplitude cost.

Both RMS = 1 mm


Radial displacement (mm)
1
0
−1
0
10
20
30
40
50
60
Fixed cyclic index
Smooth
Oscillatory

Same size, different roughness



Fourier weights of roughness

$$E_\lambda = \sum_k [1 + 4\lambda \sin^2(\pi k/n)] \|\widehat{u}_k\|^2.$$

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Higher-frequency displacement patterns receive greater penalty.

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The difference $u_{j+1} - u_j$ is not divided by arclength.

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Elastic energy along a path

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A path metric measures deformation throughout a trajectory in shape space.

Endpoint mismatch is a separate term

$$\mathcal{E}(c) + \gamma \|\mu_{c(1)} - \mu_{target}\|_K^2.$$

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The endpoint penalty promotes matching; it is not interchangeable with the path energy.

Nonoriented geometric comparison

Positions, geometric weights and tangent or normal directions define an endpoint representation.

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The implemented and contextual calculations

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The companion computes finite cyclic alignment and a quadratic displacement score. Elastic surface paths and varifold registration remain research context, with additional geometry, optimization and numerical requirements.