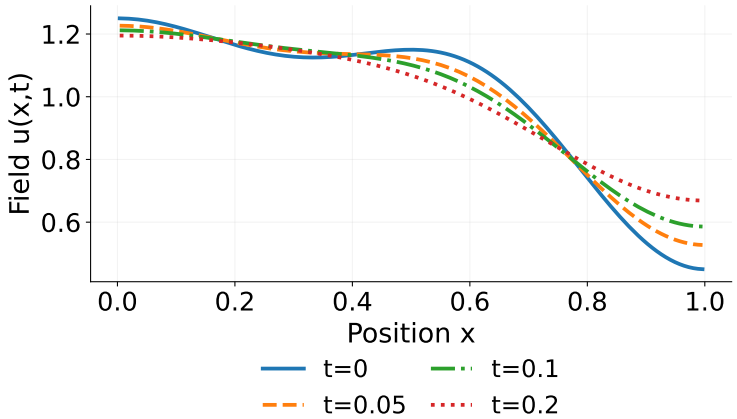
Computational Cardiology Lecture 17

Heat-flow operators, generators and finite spectral families

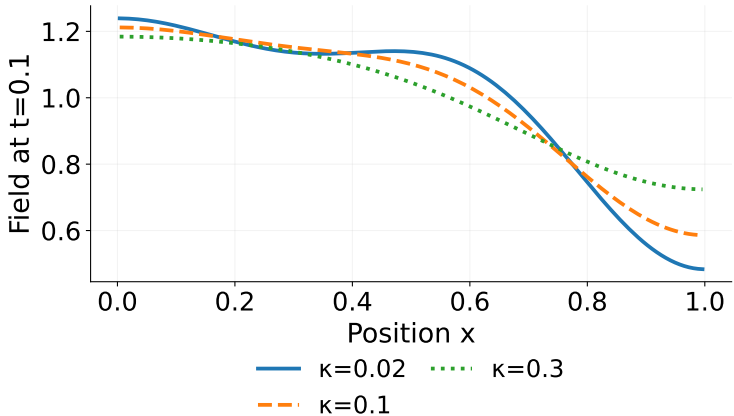
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One input $\rightarrow$ a trajectory



A parameter changes the map



A function and its representation

$$u_0 \in \mathcal{F}_K = \text{span}\{1, \cos(\pi x/L), \ldots, \cos(K\pi x/L)\}.$$

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Sampling changes the array length, not the underlying finite coefficient family.

The sealed heat equation

$$u_t = \kappa u_{xx}, \qquad u_x(0,t) = u_x(L,t) = 0.$$

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The domain, boundary conditions and diffusivity define the evolution problem.

Cosines satisfy the boundary conditions

$$\partial_x \cos(k\pi x/L) = -\frac{k\pi}{L} \sin(k\pi x/L).$$

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The derivative vanishes at both endpoints.

Separate one mode

$$u(x,t) = a_k(t)\cos(k\pi x/L) \quad \Longrightarrow \quad \dot{a}_k = -\kappa(k\pi/L)^2 a_k.$$

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Each amplitude obeys scalar exponential decay.

The exact solution operator

$$u(x,t) = a_0 + \sum_{k=1}^{K} a_k e^{-\kappa(k\pi/L)^2 t} \cos(k\pi x/L).$$

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Higher modes decay faster.

Mean preservation

$$\mathcal{M}[u] = \frac{1}{L}\int_0^L u \, \mathrm{d}x = a_0.$$

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The zero mode is unchanged even as all nonconstant modes decay.

Decay of the nonconstant energy

$$\|u - a_0\|_{L^2}^2 = \frac{L}{2} \sum_{k \geq 1} a_k^2 e^{-2\kappa(k\pi/L)^2 t}.$$

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$$\|u - a_0\|_{L^2}^2 = \frac{L}{2} \sum_{k \geq 1} a_k^2 e^{-2\kappa(k\pi/L)^2 t}.$$

For $\kappa \geq 0$, each term is nonincreasing.

The generator

$$\mathcal{A}_\kappa[u] = \kappa u_{xx}.$$

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It maps a current field to its instantaneous derivative, subject to the boundary domain.

The flow and its semigroup property

$$\Phi_t^\kappa = e^{t\mathcal{A}_\kappa}, \qquad \Phi_s^\kappa \Phi_t^\kappa = \Phi_{s+t}^\kappa.$$

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The same equation and parameters must be used in both intervals.

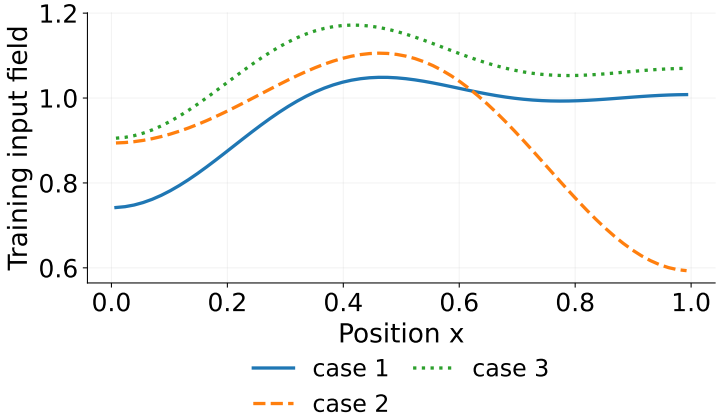
An observable is another map

A field at one time, its complete trajectory and its mean are different outputs.

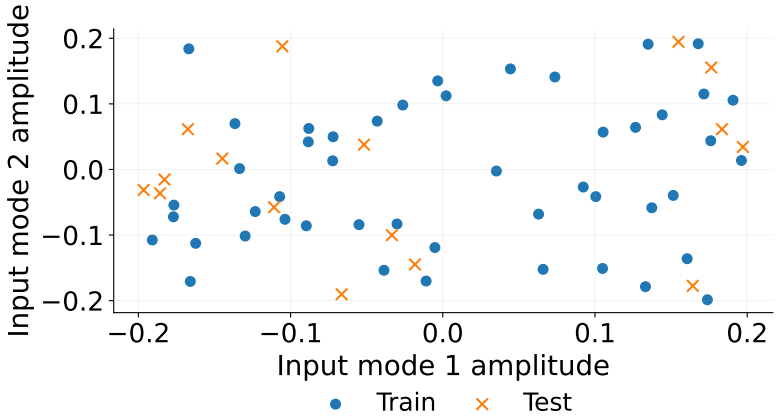
An observable is another map

A field at one time, its complete trajectory and its mean are different outputs. Correctly predicting the mean does not identify the nonconstant evolution.

Cases are complete functions



Split by case, not mesh node



Split complete evolution cases

One coefficient vector and all its snapshots remain together.

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One coefficient vector and all its snapshots remain together. Neighboring times and spatial samples from one trajectory are not new independent cases.

The discrete cosine decay rate

$$\lambda_{k,h} = \frac{4\kappa}{\Delta x^2} \sin^2\left(\frac{k\pi}{2N}\right).$$

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This tends to $\kappa(k\pi/L)^2$ for fixed k as the grid is refined.

Separate space and time error

$$e^{-t\lambda_{k,h}} \quad \text{versus} \quad e^{-t\kappa(k\pi/L)^2}$$

Separate space and time error

$$e^{-t\lambda_{k,h}} \quad \text{versus} \quad e^{-t\kappa(k\pi/L)^2}$$

isolates space;

$$(1+\Delta t\lambda_{k,h})^{-n} \quad \text{versus} \quad e^{-t\lambda_{k,h}}$$

isolates time.

Grid transfer and parameter transfer

A new grid can sample the same function family and operator.

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A new grid can sample the same function family and operator. A new diffusivity changes the attenuation itself; successful grid reuse does not establish parameter transfer.

Closed state and effective closure

The heat field is a closed autonomous state.

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The heat field is a closed autonomous state. Omitting ionic variables from a voltage evolution can instead produce a learned effective closure limited to its trajectory distribution.