Mathematical Cardiology Lecture 1

# The circulation

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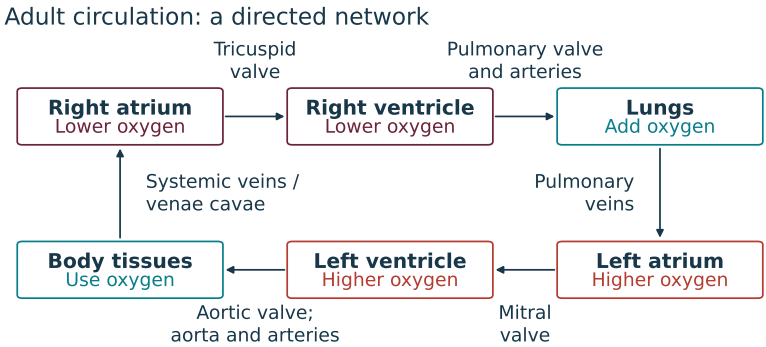
Fall 2023

# The circulation as a closed route

$$\text{left heart} \to \text{systemic bed} \to \text{right heart},$$
$$\text{right heart} \to \text{pulmonary bed} \to \text{left heart}.$$

A route specifies connections. It does not yet determine the flow on any connection.

# Chambers, valves, and vascular beds



# Chambers, valves, and vascular beds

The pulmonary and systemic beds are in series.
Within a bed, individual vessels branch and rejoin.
Arteries and veins are named by direction relative to the heart, not by oxygen content.

# Choose the quantity

| Variable | Meaning | Unit |
|---|---|---|
| $V$ | stored volume | L |
| $Q$ | volume per time | L/min |
| $P$ | pressure | mmHg |

A numerical value without its unit does not identify the quantity.

# From a rate to a passing volume

$$\Delta V_{\text{passed}} = Q\Delta t$$

$$\left(5\frac{\text{L}}{\text{min}}\right)\left(\frac{30}{60}\text{min}\right) = 2.5\text{L}.$$

This is not the net increase of volume inside the body.

# Average output per minute

$$Q = FV_s$$

$$\frac{\text{beats}}{\text{min}} \frac{\text{L}}{\text{beat}} = \frac{\text{L}}{\text{min}}.$$

A change in $F$ does not specify the change in $FV_s$ unless the change in $V_s$ is also known.

# The pressure drop

$$\Delta P = P_{\text{in}} - P_{\text{out}}$$

$$95 - 5 = 90 \text{ mmHg}, \qquad 95 - 50 = 45 \text{ mmHg}.$$

The same inlet pressure can accompany different driving differences.

# A reference shift cancels

$$\Delta P_{\text{new}} = (P_{\text{in}} + P_0) - (P_{\text{out}} + P_0).$$

$$\Delta P_{\text{new}} = P_{\text{in}} - P_{\text{out}}.$$

Changing the reference is different from physically changing the surrounding pressure of a flexible vessel.

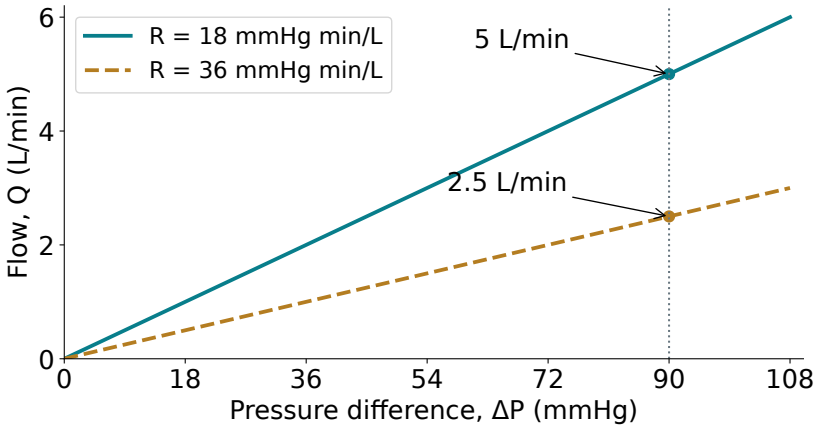
# The linear resistance law

$$\Delta P = RQ, \qquad R > 0.$$

$$Q = \frac{\Delta P}{R}, \qquad [R] = \frac{\mathrm{mmHg}}{\mathrm{L/min}}.$$

Assumptions: prescribed pressures, fixed resistance, and negligible storage within this element.

# One controlled resistance comparison



# One controlled resistance comparison

$$Q_1 = \frac{95 - 5}{18} = 5 \text{ L/min.}$$

$$Q_2 = \frac{95 - 5}{36} = 2.5 \text{ L/min.}$$

Only resistance changed. A whole-circulation response need not preserve the same pressure drop.

# Fixed pressure and fixed flow

$$\frac{Q_2}{Q_1} = \frac{\Delta P_2}{\Delta P_1}\frac{R_1}{R_2}.$$

If $\Delta P_2 = \Delta P_1$, doubling $R$ halves $Q$.
If $Q_2 = Q_1$, doubling $R$ requires twice the pressure difference.

# Sign and limiting cases

$$\Delta P = 0 \Rightarrow Q = 0.$$

$$\Delta P < 0 \Rightarrow Q < 0 \quad (R > 0).$$

A one-way valve requires an extra rule; a signed resistance alone does not enforce one-way flow.

# Count volume crossing a boundary

$$\Delta V = \Delta t \left( \sum Q_{\text{in}} - \sum Q_{\text{out}} \right).$$

A positive balance increases storage.
A zero balance is a statement about the sum of the connections.

# A branch does not duplicate flow

$$Q_0 = Q_1 + Q_2 \qquad \text{when storage does not change.}$$

For example, $5 = 2 + 3$ L/min.

Each branch can carry a different flow while the junction conserves volume.

# Internal transfers cancel

$$\Delta V_1 = (Q_{\text{in}} - q)\Delta t,$$
$$\Delta V_2 = (q - Q_{\text{out}})\Delta t.$$

$$\Delta(V_1 + V_2) = (Q_{\text{in}} - Q_{\text{out}})\Delta t.$$

# Instantaneous flow and cycle average

A chamber may store blood during part of a beat.
If its volume returns to its starting value after one complete cycle,

$$\text{volume entering per cycle} = \text{volume leaving per cycle}.$$

This is an average constraint, not equality at every instant.

# Why the loop needs a pump

$$\Delta P_{\text{pump}} - R_1 Q - \cdots - R_m Q = 0.$$

$$\Delta P_{\text{pump}} = Q \sum_i R_i.$$

With no pressure rise and positive resistances, the steady series model gives $Q = 0$.

# From one element to a circulation

The resistance calculation determines $Q$ from prescribed pressures.
A closed circulation must also determine those pressures from pumping, storage, and conservation.
The next construction adds an elastic reservoir.