Mathematical Cardiology Lecture 1

The circulation

Zan Ahmad

Fall 2023

The circulation as a closed route

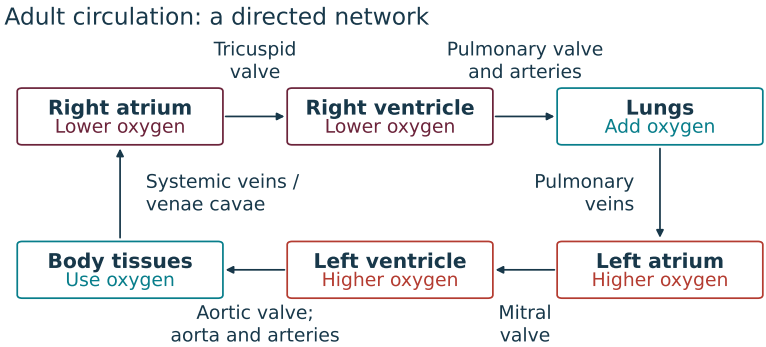
$$\text{left heart} \to \text{systemic bed} \to \text{right heart},$$
$$\text{right heart} \to \text{pulmonary bed} \to \text{left heart}.$$

The circulation as a closed route

$$\text{left heart} \to \text{systemic bed} \to \text{right heart},$$
$$\text{right heart} \to \text{pulmonary bed} \to \text{left heart}.$$

A route specifies connections. It does not yet determine the flow on any connection.

Chambers, valves, and vascular beds



Chambers, valves, and vascular beds

The pulmonary and systemic beds are in series.

Chambers, valves, and vascular beds

The pulmonary and systemic beds are in series.
Within a bed, individual vessels branch and rejoin.

Chambers, valves, and vascular beds

The pulmonary and systemic beds are in series.
Within a bed, individual vessels branch and rejoin.
Arteries and veins are named by direction relative to the heart, not by oxygen content.

Choose the quantity

Variable	Meaning	Unit
V	stored volume	L
Q	volume per time	L/min
P	pressure	mmHg

Choose the quantity

Variable	Meaning	Unit
V	stored volume	L
Q	volume per time	L/min
P	pressure	mmHg

A numerical value without its unit does not identify the quantity.

From a rate to a passing volume

$$\Delta V_{\text{passed}} = Q\Delta t$$

From a rate to a passing volume

$$\Delta V_{\text{passed}} = Q\Delta t$$

$$\left(5\frac{\text{L}}{\text{min}}\right)\left(\frac{30}{60}\text{min}\right) = 2.5\text{L}.$$

From a rate to a passing volume

$$\Delta V_{\text{passed}} = Q\Delta t$$

$$\left(5\frac{\text{L}}{\text{min}}\right)\left(\frac{30}{60}\text{min}\right) = 2.5\text{L}.$$

This is not the net increase of volume inside the body.

Average output per minute

$$Q = FV_s$$

Average output per minute

$$Q = FV_s$$

$$\frac{\text{beats}}{\text{min}} \frac{\text{L}}{\text{beat}} = \frac{\text{L}}{\text{min}}.$$

Average output per minute

$$Q = FV_s$$

$$\frac{\text{beats}}{\text{min}} \frac{\text{L}}{\text{beat}} = \frac{\text{L}}{\text{min}}.$$

A change in F does not specify the change in FV_s unless the change in V_s is also known.

The pressure drop

$$\Delta P = P_{\text{in}} - P_{\text{out}}$$

The pressure drop

$$\Delta P = P_{\text{in}} - P_{\text{out}}$$

$$95 - 5 = 90 \text{ mmHg}, \qquad 95 - 50 = 45 \text{ mmHg}.$$

The pressure drop

$$\Delta P = P_{\text{in}} - P_{\text{out}}$$

$$95 - 5 = 90 \text{ mmHg}, \qquad 95 - 50 = 45 \text{ mmHg}.$$

The same inlet pressure can accompany different driving differences.

A reference shift cancels

$$\Delta P_{\text{new}} = (P_{\text{in}} + P_0) - (P_{\text{out}} + P_0).$$

A reference shift cancels

$$\Delta P_{\text{new}} = (P_{\text{in}} + P_0) - (P_{\text{out}} + P_0).$$

$$\Delta P_{\text{new}} = P_{\text{in}} - P_{\text{out}}.$$

A reference shift cancels

$$\Delta P_{\text{new}} = (P_{\text{in}} + P_0) - (P_{\text{out}} + P_0).$$

$$\Delta P_{\text{new}} = P_{\text{in}} - P_{\text{out}}.$$

Changing the reference is different from physically changing the surrounding pressure of a flexible vessel.

The linear resistance law

$$\Delta P = RQ, \qquad R > 0.$$

The linear resistance law

$$\Delta P = RQ, \qquad R > 0.$$

$$Q = \frac{\Delta P}{R}, \qquad [R] = \frac{\mathrm{mmHg}}{\mathrm{L/min}}.$$

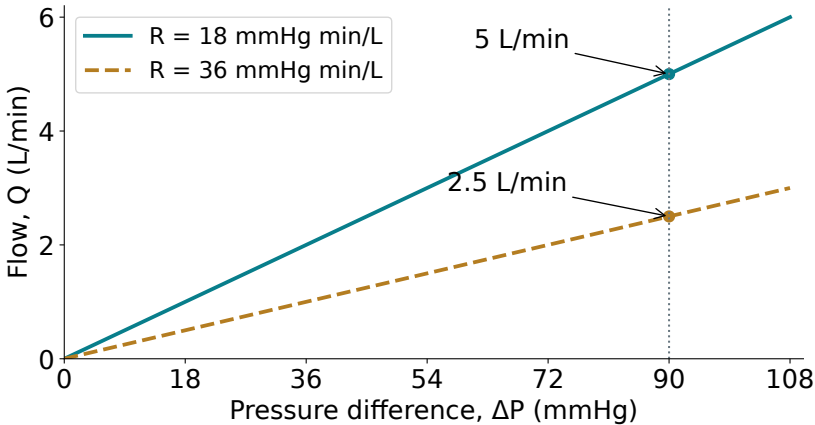
The linear resistance law

$$\Delta P = RQ, \qquad R > 0.$$

$$Q = \frac{\Delta P}{R}, \qquad [R] = \frac{\mathrm{mmHg}}{\mathrm{L/min}}.$$

Assumptions: prescribed pressures, fixed resistance, and negligible storage within this element.

One controlled resistance comparison



One controlled resistance comparison

$$Q_1 = \frac{95 - 5}{18} = 5 \text{ L/min.}$$

One controlled resistance comparison

$$Q_1 = \frac{95 - 5}{18} = 5 \text{ L/min.}$$

$$Q_2 = \frac{95 - 5}{36} = 2.5 \text{ L/min.}$$

One controlled resistance comparison

$$Q_1 = \frac{95 - 5}{18} = 5 \text{ L/min.}$$

$$Q_2 = \frac{95 - 5}{36} = 2.5 \text{ L/min.}$$

Only resistance changed. A whole-circulation response need not preserve the same pressure drop.

Fixed pressure and fixed flow

$$\frac{Q_2}{Q_1} = \frac{\Delta P_2}{\Delta P_1} \frac{R_1}{R_2}.$$

Fixed pressure and fixed flow

$$\frac{Q_2}{Q_1} = \frac{\Delta P_2}{\Delta P_1}\frac{R_1}{R_2}.$$

If $\Delta P_2 = \Delta P_1$, doubling R halves Q.

Fixed pressure and fixed flow

$$\frac{Q_2}{Q_1} = \frac{\Delta P_2}{\Delta P_1}\frac{R_1}{R_2}.$$

If $\Delta P_2 = \Delta P_1$, doubling R halves Q.
If $Q_2 = Q_1$, doubling R requires twice the pressure difference.

Sign and limiting cases

$$\Delta P = 0 \Rightarrow Q = 0.$$

Sign and limiting cases

$$\Delta P = 0 \Rightarrow Q = 0.$$

$$\Delta P < 0 \Rightarrow Q < 0 \quad (R > 0).$$

Sign and limiting cases

$$\Delta P = 0 \Rightarrow Q = 0.$$

$$\Delta P < 0 \Rightarrow Q < 0 \quad (R > 0).$$

A one-way valve requires an extra rule; a signed resistance alone does not enforce one-way flow.

Count volume crossing a boundary

$$\Delta V = \Delta t \left(\sum Q_{\text{in}} - \sum Q_{\text{out}} \right).$$

Count volume crossing a boundary

$$\Delta V = \Delta t \left(\sum Q_{\text{in}} - \sum Q_{\text{out}} \right).$$

A positive balance increases storage.

Count volume crossing a boundary

$$\Delta V = \Delta t \left(\sum Q_{\text{in}} - \sum Q_{\text{out}} \right).$$

A positive balance increases storage.
A zero balance is a statement about the sum of the connections.

A branch does not duplicate flow

$$Q_0 = Q_1 + Q_2 \qquad \text{when storage does not change.}$$

A branch does not duplicate flow

$$Q_0 = Q_1 + Q_2 \qquad \text{when storage does not change.}$$

For example, $5 = 2 + 3$ L/min.

A branch does not duplicate flow

$$Q_0 = Q_1 + Q_2 \qquad \text{when storage does not change.}$$

For example, $5 = 2 + 3$ L/min.

Each branch can carry a different flow while the junction conserves volume.

Internal transfers cancel

$$\Delta V_1 = (Q_{\text{in}} - q)\Delta t,$$
$$\Delta V_2 = (q - Q_{\text{out}})\Delta t.$$

Internal transfers cancel

$$\Delta V_1 = (Q_{\text{in}} - q)\Delta t,$$
$$\Delta V_2 = (q - Q_{\text{out}})\Delta t.$$

$$\Delta(V_1 + V_2) = (Q_{\text{in}} - Q_{\text{out}})\Delta t.$$

Instantaneous flow and cycle average

A chamber may store blood during part of a beat.

Instantaneous flow and cycle average

A chamber may store blood during part of a beat.
If its volume returns to its starting value after one complete cycle,

$$\text{volume entering per cycle} = \text{volume leaving per cycle}.$$

Instantaneous flow and cycle average

A chamber may store blood during part of a beat.
If its volume returns to its starting value after one complete cycle,

$$\text{volume entering per cycle} = \text{volume leaving per cycle}.$$

This is an average constraint, not equality at every instant.

Why the loop needs a pump

$$\Delta P_{\text{pump}} - R_1 Q - \cdots - R_m Q = 0.$$

Why the loop needs a pump

$$\Delta P_{\text{pump}} - R_1 Q - \cdots - R_m Q = 0.$$

$$\Delta P_{\text{pump}} = Q \sum_i R_i.$$

Why the loop needs a pump

$$\Delta P_{\text{pump}} - R_1 Q - \cdots - R_m Q = 0.$$

$$\Delta P_{\text{pump}} = Q \sum_i R_i.$$

With no pressure rise and positive resistances, the steady series model gives $Q = 0$.

From one element to a circulation

The resistance calculation determines Q from prescribed pressures.

From one element to a circulation

The resistance calculation determines Q from prescribed pressures.
A closed circulation must also determine those pressures from pumping, storage, and conservation.

From one element to a circulation

The resistance calculation determines Q from prescribed pressures.
A closed circulation must also determine those pressures from pumping, storage, and conservation.
The next construction adds an elastic reservoir.