Mathematical Cardiology Lecture 2

Resistance, compliance, and the reservoir equation

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Transmission and storage

$$Q = \frac{P_{\text{in}} - P_{\text{out}}}{R}, \qquad \Delta V = C \Delta P_{\text{tm}}.$$

The first equation describes transport along a pathway. The second describes expansion across a wall.

Transmural pressure

$$P_{\mathrm{tm}} = P - P_{\mathrm{ext}}.$$

Equal endpoint pressures can give zero resistive flow and nonzero vessel distension.

Compliance and its units

$$V = V_d + C(P - P_{\text{ext}}).$$

$$[C] = \text{mL/mmHg}.$$

At fixed C and P_{ext},

$$V_{n+1} - V_n = C(P_{n+1} - P_n).$$

A reservoir with a prescribed supply

$$Q_{\text{in}} = I, \qquad Q_{\text{out}} = \frac{P - P_o}{R}.$$

The reservoir pressure changes; the outlet pressure P_o is held fixed.

First write a volume balance

$$V_{n+1} - V_n \approx h(I - Q_{\text{out},n}).$$

$$C(P_{n+1} - P_n) \approx h\left(I - \frac{P_n - P_o}{R}\right).$$

The explicit pressure update

$$P_{n+1} = P_n + \frac{h}{C}\left(I - \frac{P_n - P_o}{R}\right).$$

The old pressure determines the approximate outflow during this step.

The first second

$$I = 2, \quad P_0 = P_o = 5, \quad R = 10, \quad C = 2.$$

$$Q_{\text{out},0} = 0, \qquad P_1 = 5 + \frac{1}{2}(2 - 0) = 6 \text{ mmHg}.$$

This is a one-second approximation.

Equilibrium is a flow balance

$$0 = I - \frac{P_\infty - P_o}{R}.$$

$$P_\infty = P_o + RI = 5 + 10(2) = 25 \text{ mmHg}.$$

Compliance changes the approach, not this endpoint.

Subtract equilibrium

Let $e_n = P_n - P_\infty$.

$$e_{n+1} = e_n - \frac{h}{RC} e_n = \left(1 - \frac{h}{RC}\right) e_n.$$

Repeating the same multiplication gives $e_n = (1 - h/(RC))^n e_0$.

The product RC is a time

$$[R][C] = \frac{\text{mmHg}\,\text{s}}{\text{mL}} \frac{\text{mL}}{\text{mmHg}} = \text{s}.$$

Here $RC = 10(2) = 20$ s.

A numerical step can overshoot

$$e_{n+1} = a e_n, \qquad a = 1 - h/(RC).$$

$0 \le a < 1$: monotone decay.
$-1 < a < 0$: alternating decay.
$|a| \ge 1$: no general decay.

The continuous balance

$$C\frac{dP}{dt} = I - \frac{P - P_o}{R}.$$

With $e = P - P_\infty$,

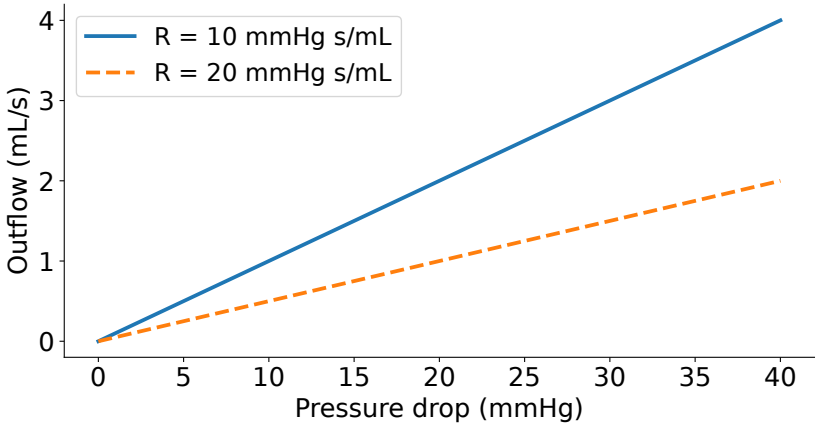
$$\frac{de}{dt} = -\frac{e}{RC}.$$

Exponential relaxation

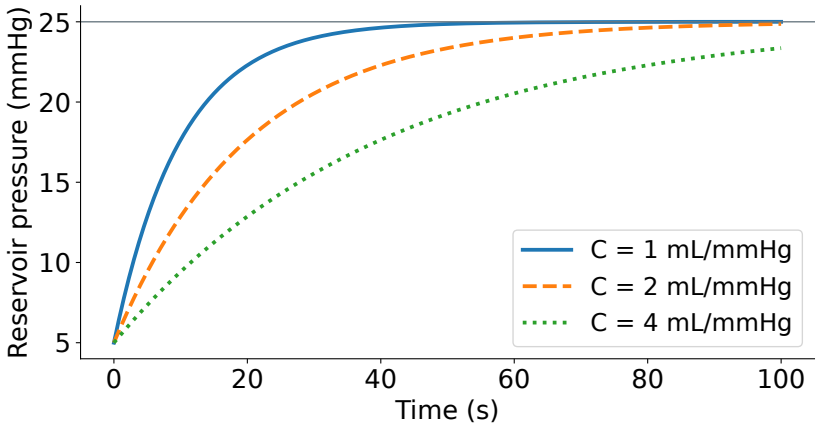
$$P(t) = P_\infty + (P_0 - P_\infty)e^{-t/(RC)}.$$

Check $t = 0$ and long time.
At $t = RC$, only $e^{-1} \approx 0.368$ of the initial gap remains.

Resistance and flow



Compliance and filling



Changing compliance

$$P_\infty = P_o + RI, \qquad \tau = RC.$$

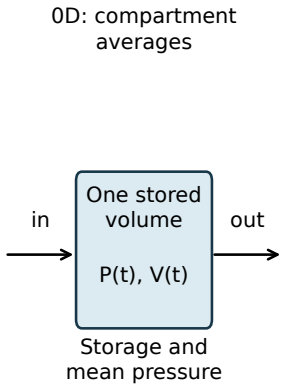
At fixed I, R, P_o, increasing C changes τ but leaves P_∞ unchanged.

Two measurements for two parameters

$$R = \frac{P_\infty - P_o}{I} \quad (I \neq 0), \qquad C = \frac{\tau}{R}.$$

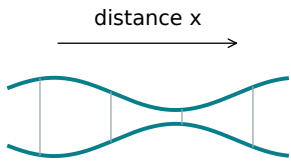
A final pressure alone leaves compliance undetermined.

Compartment storage and mean pressure



Axial variation along a vessel

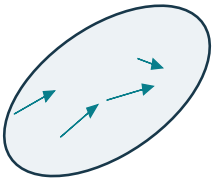
1D: along
a vessel



Travel and
reflections

Local flow and three-dimensional geometry

3D: local geometry
and flow



Jets and
wall stresses

Spatial resolution of a compartment model

0D: one compartment pressure and volume.
1D: area and flow along a vessel.
3D: a velocity field throughout local geometry.

A model can pass one check and fail another

The recurrence can conserve its volume balance while a large step gives an inaccurate trajectory.
The continuous formula can be computed exactly while the constant-compliance assumption fails experimentally.