Mathematical Cardiology Lecture 2

Resistance, compliance, and the reservoir equation

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Transmission and storage

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The first equation describes transport along a pathway. The second describes expansion across a wall.

Transmural pressure

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Equal endpoint pressures can give zero resistive flow and nonzero vessel distension.

Compliance and its units

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At fixed C and P_{ext},

$$V_{n+1} - V_n = C(P_{n+1} - P_n).$$

A reservoir with a prescribed supply

$$Q_{\text{in}} = I, \qquad Q_{\text{out}} = \frac{P - P_o}{R}.$$

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The reservoir pressure changes; the outlet pressure P_o is held fixed.

First write a volume balance

$$V_{n+1} - V_n \approx h(I - Q_{\text{out},n}).$$

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$$C(P_{n+1} - P_n) \approx h\left(I - \frac{P_n - P_o}{R}\right).$$

The explicit pressure update

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The old pressure determines the approximate outflow during this step.

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This is a one-second approximation.

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Compliance changes the approach, not this endpoint.

Subtract equilibrium

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Repeating the same multiplication gives $e_n = (1 - h/(RC))^n e_0$.

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Here $RC = 10(2) = 20$ s.

A numerical step can overshoot

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$0 \le a < 1$: monotone decay.
$-1 < a < 0$: alternating decay.
$|a| \ge 1$: no general decay.

The continuous balance

$$C\frac{dP}{dt} = I - \frac{P - P_o}{R}.$$

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$$C\frac{dP}{dt} = I - \frac{P - P_o}{R}.$$

With $e = P - P_\infty$,

$$\frac{de}{dt} = -\frac{e}{RC}.$$

Exponential relaxation

$$P(t) = P_\infty + (P_0 - P_\infty)e^{-t/(RC)}.$$

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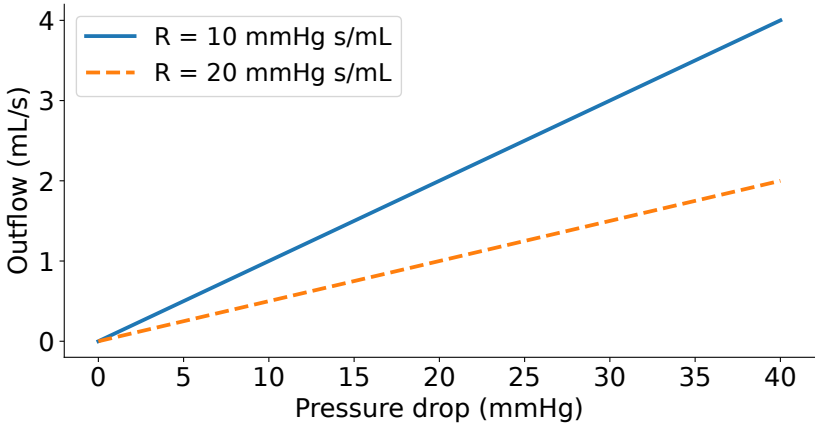
Check $t = 0$ and long time.

Exponential relaxation

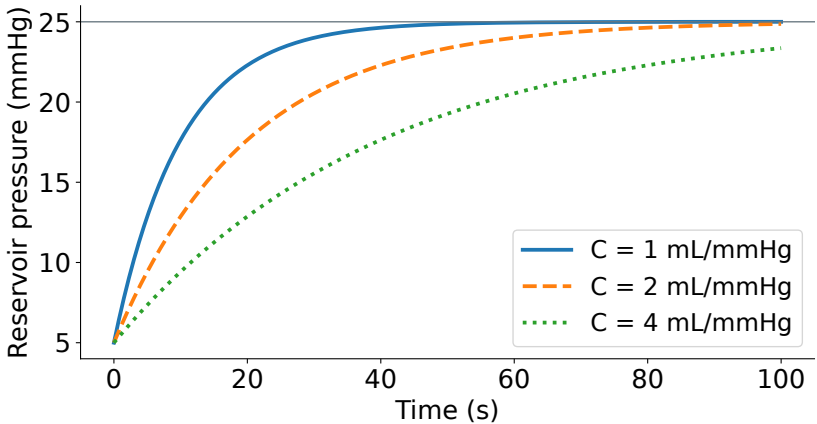
$$P(t) = P_\infty + (P_0 - P_\infty)e^{-t/(RC)}.$$

Check $t = 0$ and long time.
At $t = RC$, only $e^{-1} \approx 0.368$ of the initial gap remains.

Resistance and flow



Compliance and filling



Changing compliance

$$P_\infty = P_o + RI, \qquad \tau = RC.$$

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At fixed I, R, P_o, increasing C changes τ but leaves P_∞ unchanged.

Two measurements for two parameters

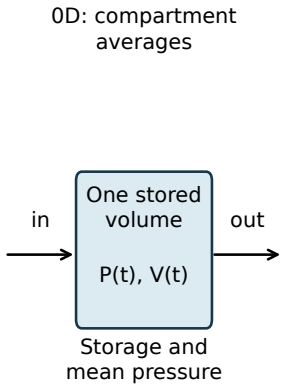
$$R = \frac{P_\infty - P_o}{I} \quad (I \neq 0), \qquad C = \frac{\tau}{R}.$$

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$$R = \frac{P_\infty - P_o}{I} \quad (I \neq 0), \qquad C = \frac{\tau}{R}.$$

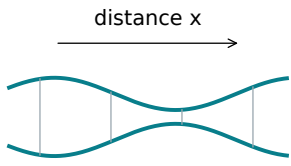
A final pressure alone leaves compliance undetermined.

Compartment storage and mean pressure



Axial variation along a vessel

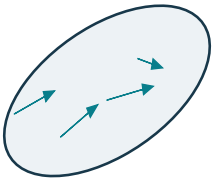
1D: along
a vessel



Travel and
reflections

Local flow and three-dimensional geometry

3D: local geometry
and flow



Jets and
wall stresses

Spatial resolution of a compartment model

0D: one compartment pressure and volume.

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1D: area and flow along a vessel.

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0D: one compartment pressure and volume.
1D: area and flow along a vessel.
3D: a velocity field throughout local geometry.

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The recurrence can conserve its volume balance while a large step gives an inaccurate trajectory.

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The recurrence can conserve its volume balance while a large step gives an inaccurate trajectory.
The continuous formula can be computed exactly while the constant-compliance assumption fails experimentally.