Mathematical Cardiology Lecture 4

# Membrane voltage and a traveling activation

Zan Ahmad

Fall 2023

# Three different observations

$V_m(t)$: one local membrane trace.
$t_a(x)$: an arrival time for each location.
ECG: a body-surface measurement of distributed activity.

# The sign of membrane voltage

$$V_m = \phi_{\text{inside}} - \phi_{\text{outside}}.$$

$V_m < 0$ means lower internal electric potential under this convention.

# Capacitive charge balance

$$C_m \Delta V_m \approx h \left( I_{\text{applied}} - \sum_j I_j \right).$$

Outward ionic current is positive here.
Dividing by $h$ gives a rate of change.

# A current depends on a driving difference

$$I_j = g_j(V_m - E_j).$$

At $V_m = E_j$, this channel current is zero.
Positive voltage alone does not determine current direction.

# Temporarily freeze the conductances

With $I_{\text{applied}} = 0$ and $g = \sum_j g_j > 0$,

$$\overline{E} = \frac{\sum_j g_j E_j}{g}.$$

$$C_m \frac{dV_m}{dt} = -g(V_m - \overline{E}).$$

Time scale: $C_m/g$.

# Count gates available to change

Closed fraction: $1 - n$; open fraction: $n$.

$$\Delta n \approx h\{\alpha(1 - n) - \beta n\}.$$

At equilibrium, $n = \alpha/(\alpha + \beta)$.

# Where independence enters

Four identical independent gates must all be open.

$$\Pr(\text{channel open}) = n \cdot n \cdot n \cdot n = n^4.$$

This is a nerve-model example, not the cardiac companion calculation.

# The supplied waveform

| $a$ | $0$ | $2$ | $12$ | $160$ | $220$ |
|---|---|---|---|---|---|
| $F(a)$ | $-85$ | $30$ | $15$ | $0$ | $-85$ |

Ages are ms; voltages are mV. Straight segments connect the listed points.

# Interpolation within one segment

$$F(a) = F_i + \frac{a - a_i}{a_{i+1} - a_i}(F_{i+1} - F_i).$$

The fraction is zero at the first endpoint and one at the second.

# A travel time

$$t_a(x) = \frac{x}{c}.$$

$$\frac{20 \text{ mm}}{0.4 \text{ mm/ms}} = 50 \text{ ms}.$$

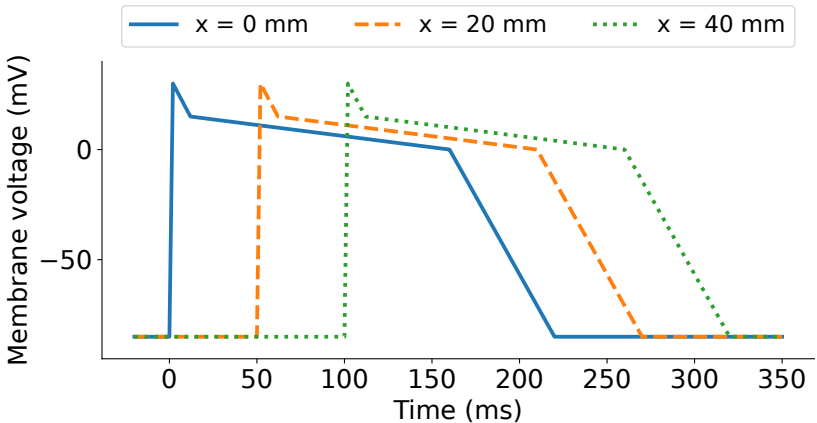
# Evaluate the local age

$$a = t - t_a(x) = t - x/c.$$

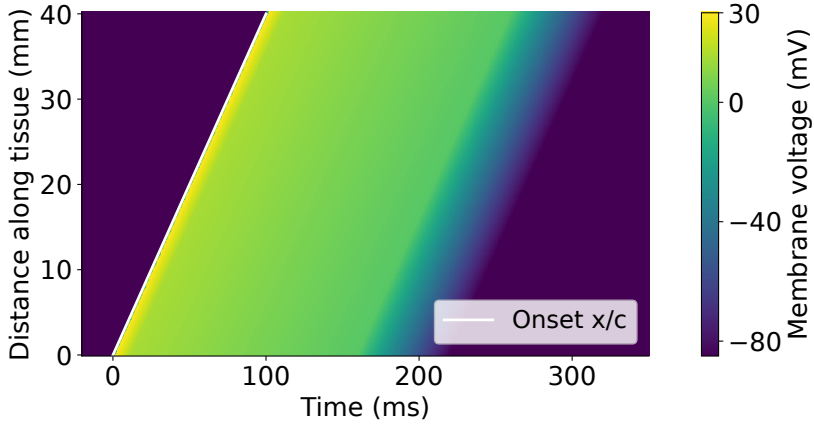
At $t = 80$ ms and $x = 20$ mm, $a = 30$ ms.

$$V_m(x, t) = F(a).$$

# Delayed local membrane-voltage traces



# Space-time voltage at c = 0.4 mm/ms



# One shape at several locations

$$V_m(x,t) = F(t - x/c).$$

Position changes delay only. The amplitude and duration remain prescribed.

# Read the slope on the actual axes

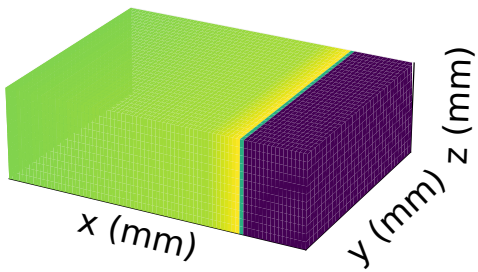
For fixed local age $a_0$,

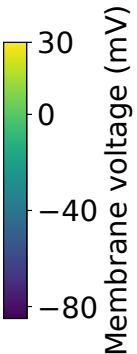
$$x = c(t - a_0).$$

With time horizontal and distance vertical, the phase-line slope is $c$.

# Prescribed activation at 80 ms: c = 0.4 mm/ms

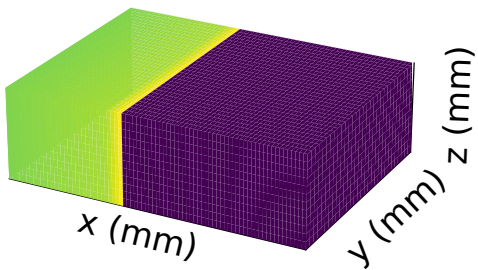
40 × 30 × 8 mm slab; onset at 32 mm

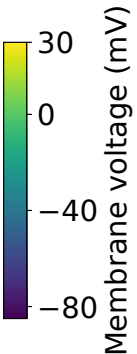


# Prescribed activation at 80 ms: c = 0.2 mm/ms

40 × 30 × 8 mm slab; onset at 16 mm



# The slab inherits an imposed symmetry

$$V_m(x, y, z, t) = F(t - x/c).$$

No $y$ or $z$ dependence is solved.

$$ct = 32 \text{ mm or } 16 \text{ mm} \quad (t = 80 \text{ ms}).$$

# A waveform is not an excitability model

The construction has no stimulus, gate recovery or computed conduction speed.
It therefore cannot predict premature-stimulus block or reentry.
Multiple-location recordings can test the constant-speed assumption.