Mathematical Cardiology Lecture 4

Membrane voltage and a traveling activation

Zan Ahmad

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Three different observations

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ECG: a body-surface measurement of distributed activity.

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$V_m < 0$ means lower internal electric potential under this convention.

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Dividing by h gives a rate of change.

A current depends on a driving difference

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Positive voltage alone does not determine current direction.

Temporarily freeze the conductances

With $I_{\text{applied}} = 0$ and $g = \sum_j g_j > 0$,

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Time scale: C_m/g.

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At equilibrium, $n = \alpha/(\alpha + \beta)$.

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This is a nerve-model example, not the cardiac companion calculation.

The supplied waveform

a	0	2	12	160	220
$F(a)$	-85	30	15	0	-85

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Ages are ms; voltages are mV. Straight segments connect the listed points.

Interpolation within one segment

$$F(a) = F_i + \frac{a - a_i}{a_{i+1} - a_i}(F_{i+1} - F_i).$$

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The fraction is zero at the first endpoint and one at the second.

A travel time

$$t_a(x) = \frac{x}{c}.$$

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$$\frac{20 \text{ mm}}{0.4 \text{ mm/ms}} = 50 \text{ ms}.$$

Evaluate the local age

$$a = t - t_a(x) = t - x/c.$$

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At $t = 80$ ms and $x = 20$ mm, $a = 30$ ms.

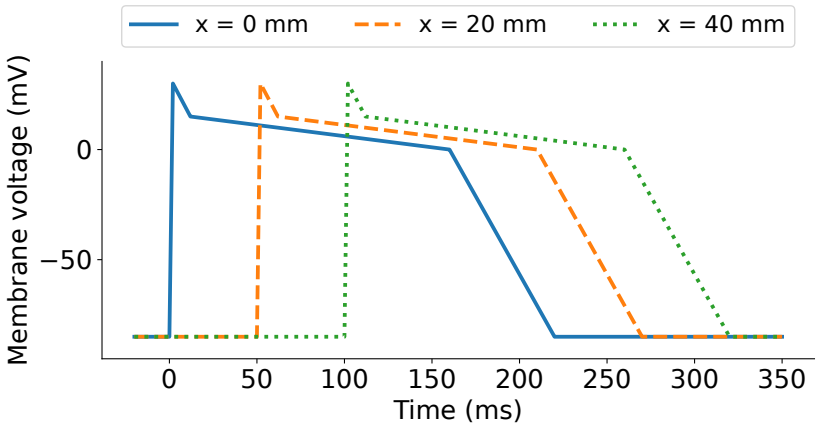
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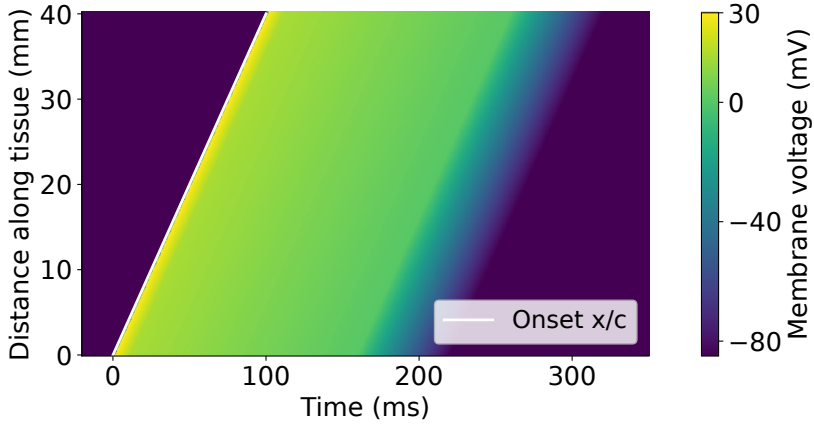
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$$V_m(x, t) = F(a).$$

Delayed local membrane-voltage traces



Space-time voltage at c = 0.4 mm/ms



One shape at several locations

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Position changes delay only. The amplitude and duration remain prescribed.

Read the slope on the actual axes

For fixed local age a_0,

$$x = c(t - a_0).$$

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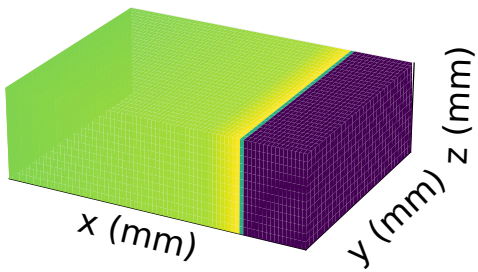
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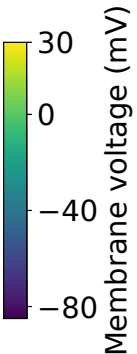
$$x = c(t - a_0).$$

With time horizontal and distance vertical, the phase-line slope is c.

Prescribed activation at 80 ms: c = 0.4 mm/ms

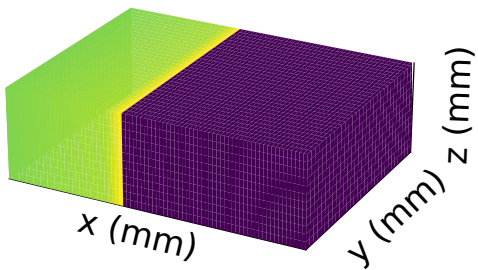
40 × 30 × 8 mm slab; onset at 32 mm

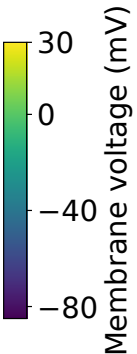




Prescribed activation at 80 ms: c = 0.2 mm/ms

40 × 30 × 8 mm slab; onset at 16 mm





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$$ct = 32 \text{ mm or } 16 \text{ mm} \quad (t = 80 \text{ ms}).$$

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Multiple-location recordings can test the constant-speed assumption.