Mathematical Cardiology Lecture 5

Parallel circulation, shunt flow, and oxygen delivery

Lecture notes

Changing a connection changes the equations of a circulation. In a serial route the same sustained flow passes through successive beds. In a parallel route a supplied flow divides between beds sharing endpoint pressures. We shall derive that division, then add an oxygen balance. The resulting calculation shows why a larger oxygen content need not imply a larger oxygen delivery rate.

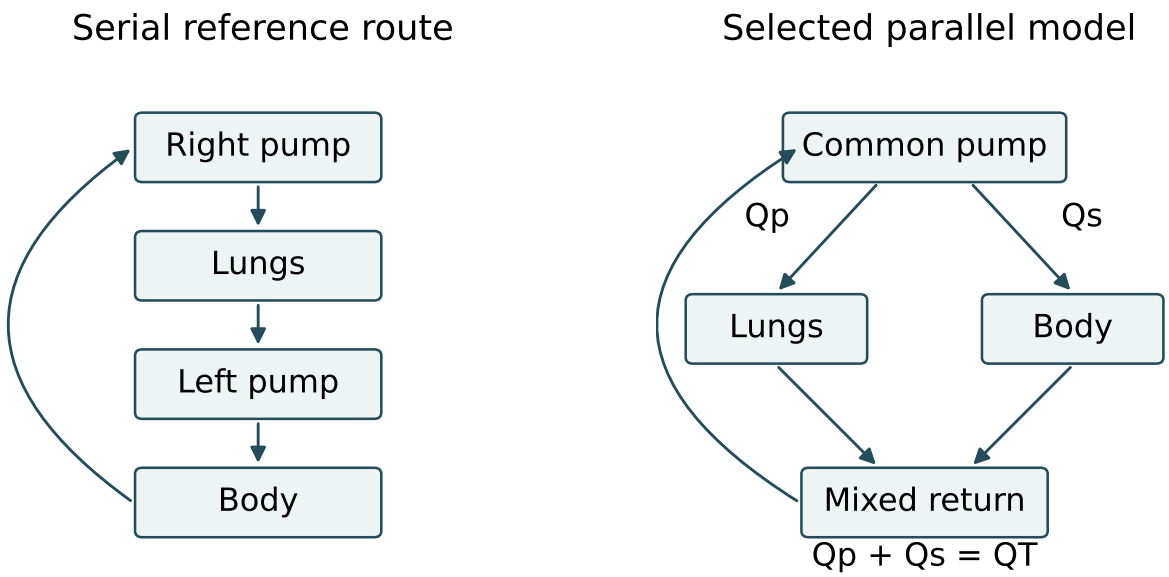
1 Topology comes before the flow calculation

Slides for this section

The ordinary adult route places systemic and pulmonary beds in series between the two sides of the heart. In a steady state without shunts or accumulation, their average flows agree. A selected single-pump abstraction instead places the two beds in parallel and completely mixes their returns. The two pictures are not interchangeable.

A congenital difference may change a connection, obstruct a route or alter effective pumping. A single functional ventricle describes multiple anatomical situations. The parallel model below is a deliberate abstraction, not a model of a completed Fontan circulation. Fontan routing carries systemic venous return toward the pulmonary arteries without a separate subpulmonary pumping ventricle. The network must be specified before a number is given a clinical interpretation.

For one extra passive connection, choose positive flow from node a to node b. Then $Q_h = (P_a - P_b)/R_h$, with $R_h > 0$. For pressures 12 and 8 mmHg and resistance 2 mmHg min/L, $Q_h = 2$ L/min. Exchanging the pressures gives -2 L/min. A negative answer reverses direction; it does not represent a negative volume of blood. Opening an actual connection may also change both pressures, which this prescribed-pressure calculation does not predict.



Original route diagrams. The single-pump complete-mixing network is a selected classroom abstraction and is not a Fontan circulation. Arrows show routes, not blood color.

2 Solve the parallel-flow equations

Slides for this section

Let $Q_T > 0$ be a prescribed total output. The pulmonary and systemic branches have positive resistances R_p, R_s and share the same pressure difference ΔP. Their equations are

$$Q_p = \frac{\Delta P}{R_p}, \qquad Q_s = \frac{\Delta P}{R_s}, \qquad Q_p + Q_s = Q_T.$$

Substitution into conservation gives

$$Q_T = \Delta P \left(\frac{1}{R_p} + \frac{1}{R_s} \right), \qquad \Delta P = Q_T \frac{R_p R_s}{R_p + R_s}.$$

Dividing by each branch resistance yields

$$Q_p = Q_T \frac{R_s}{R_p + R_s}, \qquad Q_s = Q_T \frac{R_p}{R_p + R_s}. \tag{2.1}$$

The opposite resistance appears in each numerator because both branch laws and the total-flow constraint have been used. Equivalently, flow divides in proportion to conductance $G = 1/R$.

With $Q_T = 5$ L/min, $R_p = 10$ and $R_s = 15$ mmHg min/L, the common drop is 30 mmHg, $Q_p = 3$ L/min and $Q_s = 2$ L/min. Both products $R_p Q_p$ and $R_s Q_s$ recover the same drop. Equal resistances give equal shares; a very large pulmonary resistance sends its share toward zero. Closing a branch is represented by zero conductance, not zero resistance.

3 Mix contents by conserving oxygen

Slides for this section

Let C_p, C_s be oxygen contents of the returning streams, measured in mL of oxygen per liter of blood. Content is neither a partial pressure nor a saturation percentage. We prescribe $C_p = 200$ and $C_s = 120$ in these units; no gas-exchange or hemoglobin model calculates them here.

Over an interval Δt, incoming oxygen volumes are $Q_p C_p \Delta t$ and $Q_s C_s \Delta t$. Complete mixing produces a common outgoing content C_m. Equal incoming and outgoing oxygen volumes require

$$Q_T C_m \Delta t = Q_p C_p \Delta t + Q_s C_s \Delta t.$$

Cancel the duration and divide by the positive total flow:

$$C_m = \frac{Q_p C_p + Q_s C_s}{Q_T}. \tag{3.1}$$

For the worked split, the oxygen-volume rates are 600 and 240 mL/min, and $C_m = 840/5 = 168$ mL/L. The mixture must lie between the two contents because the nonnegative weights Q_p/Q_T and Q_s/Q_T add to one. At zero total incoming flow, this mixing equation cannot identify a concentration.

4 Content and delivery enter as a product

Slides for this section

Systemic delivery is the oxygen-volume rate entering the systemic bed:

$$D_s = Q_s C_m. \tag{4.1}$$

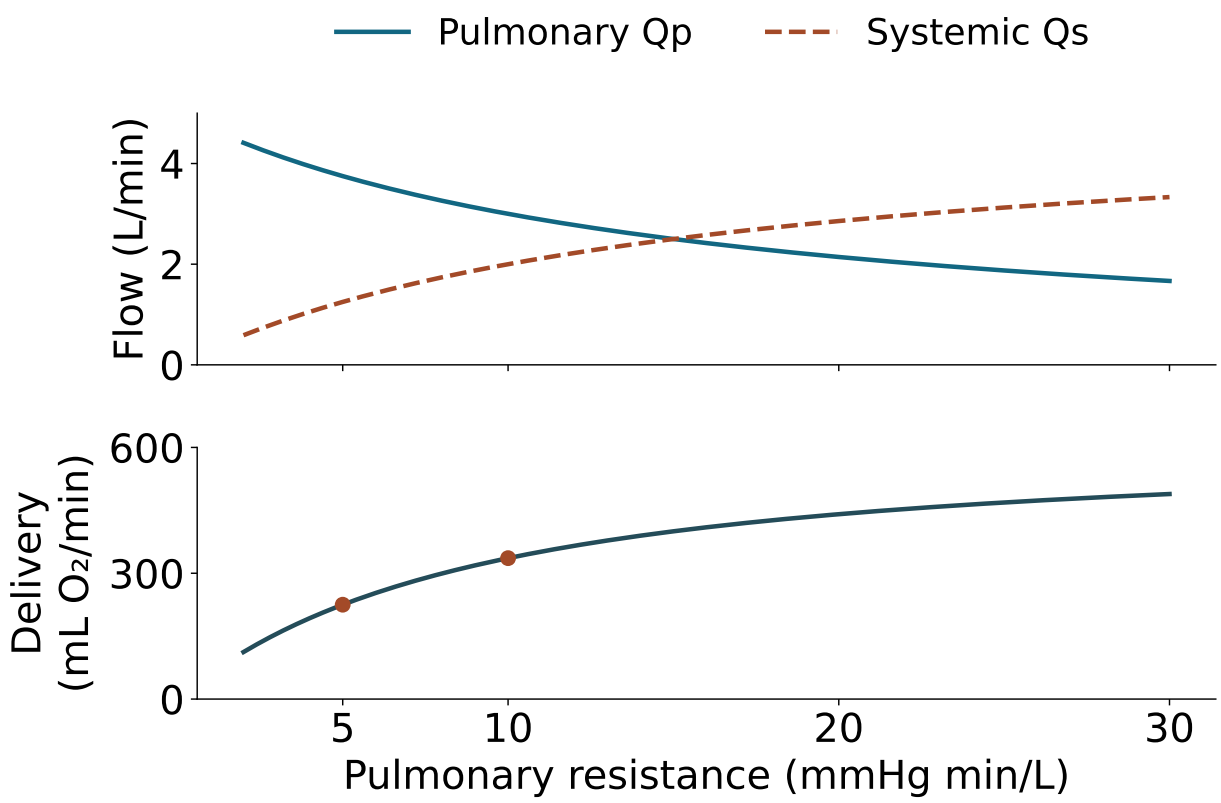
In the first example it is $2(168) = 336$ mL oxygen/min. This is delivery, not tissue consumption; consumption would also subtract the outgoing oxygen rate and require a consistent metabolic balance.

Now lower only R_p from 10 to 5, preserving total output, systemic resistance and both return contents. The formulas give $Q_p = 3.75$, $Q_s = 1.25$ L/min and $C_m = 180$ mL/L. Content rises, but delivery falls to 225 mL/min.

The relationship can be made explicit by setting $x = Q_p/Q_T$. Then $Q_s = Q_T(1 - x)$ and

$$C_m = C_s + x(C_p - C_s), \tag{4.2}$$
$$D_s = Q_T(1 - x)\{C_s + x(C_p - C_s)\}. \tag{4.3}$$

For the chosen contents, $D_s/Q_T = 120 - 40x - 80x^2$. If x increases by a positive amount δ while remaining in $[0, 1]$, the change is $-40\delta - 160x\delta - 80\delta^2 < 0$. Thus increasing the pulmonary fraction decreases systemic delivery throughout this particular fixed-content example, even while mixed content increases. This conclusion follows from the declared constants; it is not a universal rule for a physiological circulation.



QT = 5 L/min, Rs = 15 mmHg min/L, Cp = 200 and Cs = 120 mL O2/L. Only Rp changes; contents and output remain prescribed.

5 Streaming, extraction, and missing oxygen dynamics

Slides for this section

Complete mixing is a specific assumption. Preferential streaming can make different outgoing branches receive different contents. Prescribing the return contents while changing flow also omits changes in extraction, ventilation and gas exchange. A closed oxygen-metabolism model would have to calculate those quantities rather than hold them fixed.

The branch laws, total-flow balance and oxygen balance can each be tested separately. A numerical result should satisfy $Q_p + Q_s = Q_T$, equal branch pressure drops, and equality of incoming and outgoing oxygen-volume rates. Passing those tests establishes consistency of the abstraction. It cannot identify an appropriate patient-specific resistance, surgical outcome or target saturation. A focused project can instead state the chosen network and compare which conclusion survives when one of its fixed quantities becomes a modeled variable.

Exercises

MC05-E1. A signed connection

A passive shunt has a positive arrow from A to B and resistance 4 mmHg min/L.
a For PA = 14 and PB = 6 mmHg, find flow and direction.

b Increase both pressures by 5 mmHg. Recompute flow and explain.

c Set PA = 6 and PB = 14. Find signed flow and explain why a resistor may not represent a valve.

MC05-E2. Check parallel flow and mixing

Use QT = 6 L/min, Rp = 20 and Rs = 10 mmHg min/L. Prescribe Cp = 210 and Cs = 120 mL O2/L; assume complete mixing.
a Find Qp and Qs and verify the common pressure drop.

b Find mixed content, showing the oxygen-volume balance.

c Find systemic delivery and name two biological processes omitted by prescribing return contents.

MC05-E3. Challenge an attractive conclusion

A report says: "Our resistance change raised the mixed oxygen content, therefore it improved systemic oxygen delivery."
a Give a numerical counterexample using two positive flows and contents of your choosing.

b State a bounded project question and two quantities to hold fixed.

c Name one model check and one independent observation needed before interpreting the model physiologically.

References

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