Mathematical Cardiology Lecture 5

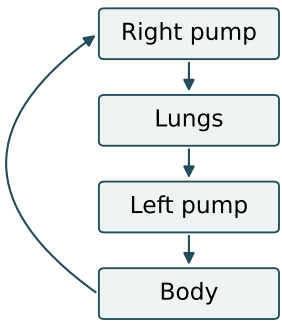
Parallel circulation, shunt flow, and oxygen delivery

Zan Ahmad

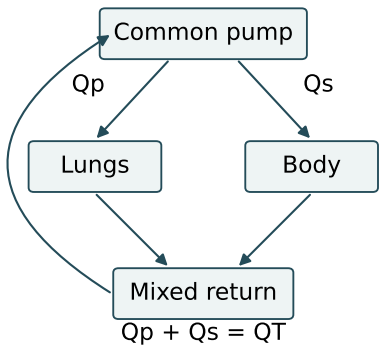
Fall 2023

Serial and parallel routes

Serial reference route



Selected parallel model



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Parallel: the total flow divides among branches.
The selected parallel abstraction is not a Fontan model.

A shunt sign convention

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The chosen arrow defines positive flow.

Three equations for a parallel split

$$Q_p = \Delta P / R_p, \qquad Q_s = \Delta P / R_s.$$

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$$Q_p + Q_s = Q_T.$$

Eliminate the branch flows

$$Q_T = \Delta P \left(\frac{1}{R_p} + \frac{1}{R_s} \right).$$

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$$Q_T = \Delta P \left(\frac{1}{R_p} + \frac{1}{R_s} \right).$$

$$\Delta P = Q_T \frac{R_p R_s}{R_p + R_s}.$$

The opposite resistance in the numerator

$$Q_p = Q_T \frac{R_s}{R_p + R_s}, \qquad Q_s = Q_T \frac{R_p}{R_p + R_s}.$$

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$$Q_p = Q_T \frac{R_s}{R_p + R_s}, \qquad Q_s = Q_T \frac{R_p}{R_p + R_s}.$$

$$\frac{Q_p}{Q_s} = \frac{R_s}{R_p}.$$

Check both laws and conservation

$$Q_T = 5, \quad R_p = 10, \quad R_s = 15.$$

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$$3 + 2 = 5, \qquad 10(3) = 15(2) = 30.$$

Oxygen content has a transport unit

$$[C] = \text{mL } O_2/\text{L blood}.$$

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Content is not saturation or partial pressure.

Mixing is an oxygen balance

$$Q_T C_m \Delta t = Q_p C_p \Delta t + Q_s C_s \Delta t.$$

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$$Q_T C_m \Delta t = Q_p C_p \Delta t + Q_s C_s \Delta t.$$

$$C_m = \frac{Q_p C_p + Q_s C_s}{Q_T}.$$

The numerical mixture

$$Q_p C_p = 3(200) = 600, \quad Q_s C_s = 2(120) = 240.$$

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$$Q_pC_p = 3(200) = 600, \quad Q_sC_s = 2(120) = 240.$$

$$C_m = \frac{600 + 240}{5} = 168 \text{ mL } O_2/\text{L}.$$

Why the mixture lies between its inputs

Let $x = Q_p/Q_T$, with $0 \leq x \leq 1$.

Why the mixture lies between its inputs

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$$C_m = xC_p + (1 - x)C_s.$$

Why the mixture lies between its inputs

Let $x = Q_p/Q_T$, with $0 \le x \le 1$.

$$C_m = xC_p + (1-x)C_s.$$

The weights are nonnegative and sum to one.

Delivery uses systemic flow

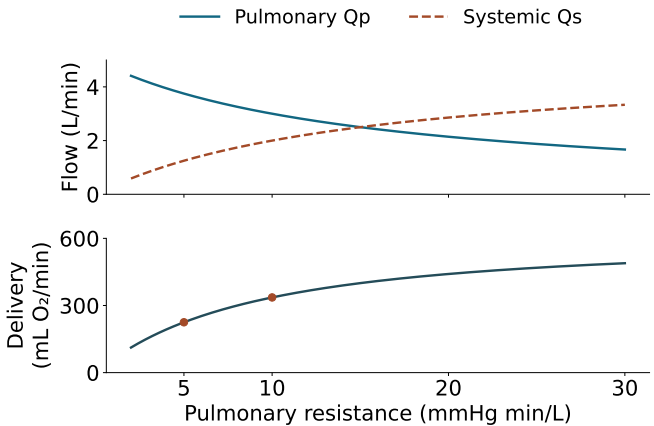
$$D_s = Q_s C_m = 2(168) = 336 \text{ mL } O_2/\text{min}.$$

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$$D_s = Q_s C_m = 2(168) = 336 \text{ mL } O_2/\text{min}.$$

Delivery entering a bed differs from consumption within it.

Lower only pulmonary resistance



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$$C_m = 180, \qquad D_s = 1.25(180) = 225.$$

Lower only pulmonary resistance

$$R_p : 10 \to 5 \Rightarrow Q_p = 3.75,\ Q_s = 1.25.$$

$$C_m = 180, \qquad D_s = 1.25(180) = 225.$$

Higher mixed content accompanies lower systemic delivery.

Write the tradeoff as one polynomial

$$\frac{D_s}{Q_T} = (1-x)\{120 + 80x\}.$$

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$$= 120 - 40x - 80x^2.$$

Compare two pulmonary fractions

For $\delta > 0$ and $x + \delta \leq 1$,

$$\frac{D_s(x+\delta) - D_s(x)}{Q_T} = -40\delta - 160x\delta - 80\delta^2 < 0.$$

Compare two pulmonary fractions

For $\delta > 0$ and $x + \delta \leq 1$,

$$\frac{D_s(x+\delta) - D_s(x)}{Q_T} = -40\delta - 160x\delta - 80\delta^2 < 0.$$

This conclusion uses the prescribed return contents.

Close the accounting, then challenge the assumptions

Check equal branch drops, total blood flow, and oxygen-volume balance.

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Check equal branch drops, total blood flow, and oxygen-volume balance.
Then ask whether complete mixing and fixed return contents describe the intended system.