Mathematical Cardiology Lecture 6

Feedback, stability, and hydrostatic pressure

Zan Ahmad

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Pressure as a function of rate

With fixed stroke volume,

$$Q = HV_s/1000.$$

With one downstream resistance,

$$P = P_v + RQ = P_v + aH, \qquad a = RV_s/1000.$$

H is in beats/min and Q is in L/min.

The initial pressure

$$V_s = 100, \quad P_v = 5, \quad R = 15, \quad H = 60.$$

$$Q = 6\,\mathrm{L/min}, \qquad P = 5 + 15(6) = 95\,\mathrm{mmHg}.$$

The equation describes a mean pressure; it contains no pulse or storage state.

A resistance disturbance

Hold $H = 60$ and let R fall from 15 to 10.

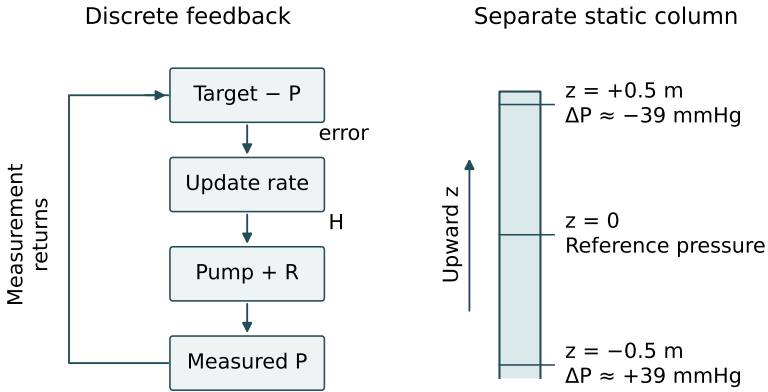
$$P = 5 + 10(6) = 65\,\mathrm{mmHg}.$$

For a $95\,\mathrm{mmHg}$ target,

$$P_* - P = 30\,\mathrm{mmHg}.$$

This error is the input to the next correction.

A measured error changes the next rate



A measured error changes the next rate

$$H_{n+1} = H_n + k(P_* - P_n), \qquad P_n = P_v + aH_n.$$

Use $k = 0.5$ once every ten seconds.

$$H_1 = 60 + 0.5(95 - 65) = 75\,\text{beats/min}.$$

Then $P_1 = 5 + 75 = 80\,\text{mmHg}$.

Continue the recurrence

At $R = 10$, $P_n = 5 + H_n$.

n	0	1	2	3
H_n	60	75	82.5	86.25
P_n	65	80	87.5	91.25

Each pressure error is half the preceding error. Why?

Find the equilibrium first

At equilibrium, $H_{n+1} = H_n = H_*$.

$$0 = k(P_* - P_v - aH_*).$$

For $k \neq 0$ and $a > 0$,

$$H_* = (P_* - P_v)/a.$$

Here $H_* = 90\,\text{beats/min}$.

Subtract the equilibrium

Put $e_n = H_n - H_*$. Since $P_* - P_v = aH_*$,

$$e_{n+1} = e_n - kae_n.$$

Thus

$$\boxed{e_{n+1} = (1-ka)e_n}, \qquad e_n = (1-ka)^n e_0.$$

Read stability from one multiplier

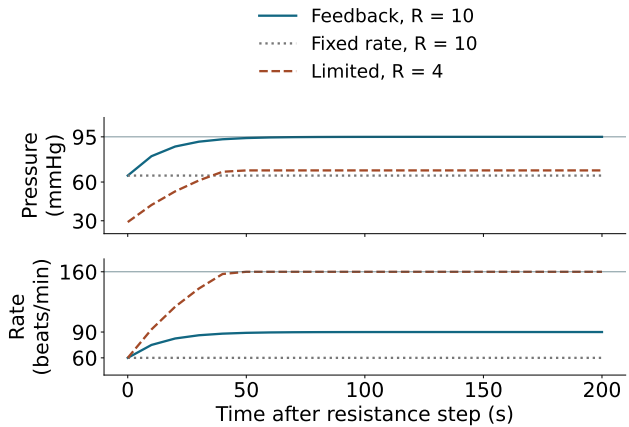
Let $r = 1 - ka$.

$$e_n = r^n e_0.$$

$0 < ka < 1$	same-sign shrinking errors
$ka = 1$	one-step correction
$1 < ka < 2$	alternating shrinking errors
$ka = 2$	persistent alternation

Convergence requires $|r| < 1$.

The actual pressure sequence



The actual pressure sequence

For $ka = 0.5$, $H_0 - H_* = -30$.

$$H_n = 90 - 30(1/2)^n, \qquad P_n = 95 - 30(1/2)^n.$$

Each factor of $1/2$ takes ten seconds in this experiment.

The rate cannot increase without bound

The implementation clips the proposed rate to $[40, 160]$. At $R = 4$,

$$H_* = (95 - 5)/0.4 = 225 > 160.$$

At the upper bound,

$$P = 5 + 0.4(160) = 69\,\mathrm{mmHg}.$$

Why more gain does not solve this case

The largest available flow is

$$Q_{\max} = 160(100)/1000 = 16\,\mathrm{L/min}.$$

The largest available pressure is $69\,\mathrm{mmHg}$. The residual error is $95 - 69 = 26\,\mathrm{mmHg}$. Changing k cannot change this bound.

Physiological feedback mechanisms

The recurrence has one sensor, one scalar error and one actuator. A baroreflex can alter vascular and cardiac behavior, including mechanisms absent from $P = P_v + aH$.
Delays and additional states can change the multiplier calculation.

The weight of a static column

A column of depth d and area A has weight $\rho A d g$.

$$(P_{\text{bottom}} - P_{\text{top}})A = \rho A d g.$$

$$\boxed{P_{\text{bottom}} - P_{\text{top}} = \rho g d}.$$

Half a meter below the reference

$$\Delta P = 1060(9.81)(0.5) = 5199.3\,\mathrm{Pa}.$$

$$\Delta P = 5199.3/133.322 \simeq 39.0\,\mathrm{mmHg}.$$

If z increases upward, $\Delta P = -\rho g \Delta z$.

A column is not an orthostatic model

Around a loop returning to its starting height,

$$\sum \Delta z = 0 \quad \Longrightarrow \quad \sum(-\rho g \Delta z) = 0.$$

The displayed controller contains no height input, venous pooling, or posture-dependent stroke volume.