Mathematical Cardiology Lecture 8

Calibrating a flow law and constructing a classifier

Lecture notes

A physical relation can be known in form while containing an unknown parameter. In $Q = G\Delta P$, for example, a measured flow can tell us something about conductance G. The same observation cannot also serve as an independent test of the parameter it selected. We shall first derive a calibration rule, then construct a classifier whose fitted quantities are equally explicit. In both cases the algebra will reveal what the data can determine and what must remain outside the fitted calculation.

1 A passive law with an unknown coefficient

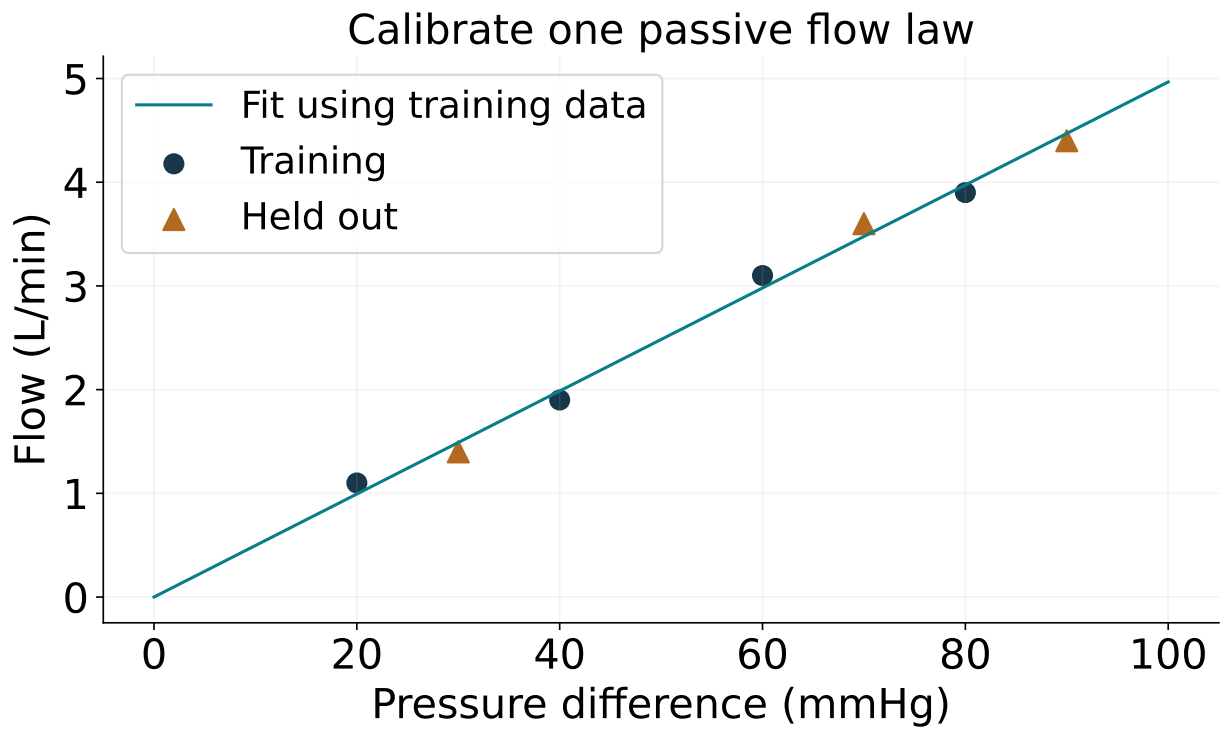
Slides for this section

Let x_i be the pressure difference in experiment i, measured in mmHg, and let y_i be the observed flow in L/min. The proposed law predicts $\widehat{y}_i = Gx_i$, where G has units $\mathrm{L}/(\mathrm{min}\,\mathrm{mmHg})$. Its resistance is $R = 1/G$ when $G > 0$. Choosing a line through the origin is a physical assumption: equal boundary pressures produce zero passive flow. An offset would require a different model, perhaps one for measurement bias or an active pressure source.

Define the residual as prediction minus observation, $r_i(G) = Gx_i - y_i$. Positive and negative residuals can cancel even when both are large. We therefore consider

$$S(G) = \sum_{i \in \mathcal{T}} (Gx_i - y_i)^2, \tag{1.1}$$

where $\mathcal{T}$ contains training observations only. Each term and the sum have units $(\mathrm{L/min})^2$. Squaring is a choice of discrepancy measure; it gives a larger penalty to a large error than to several smaller errors with the same total absolute size. Neither this choice nor the resulting minimizer proves that the passive law is the right mechanism.



Synthetic training and held-out observations with a fitted through-origin law. The line predicts flow from a prescribed pressure difference; it does not simulate a complete circulation.

2 Least squares without differentiation

Slides for this section

Expand the square and collect the terms involving G:

$$S(G) = AG^2 - 2BG + C, \qquad A = \sum_{\mathcal{T}} x_i^2, \quad B = \sum_{\mathcal{T}} x_i y_i, \quad C = \sum_{\mathcal{T}} y_i^2. \tag{2.1}$$

If $A > 0$, completing the square gives

$$S(G) = A\left(G - \frac{B}{A}\right)^2 + C - \frac{B^2}{A}. \tag{2.2}$$

The second and third terms do not depend on G, and the squared term is nonnegative. Thus the unique unconstrained minimizer is

$$\widehat{G} = \frac{\sum_{\mathcal{T}} x_i y_i}{\sum_{\mathcal{T}} x_i^2}. \tag{2.3}$$

This argument uses only expansion and a nonnegative square. If every $x_i = 0$, then $A = 0$ and $S(G) = C$ for every conductance: the experiment contains no information about the slope. The companion additionally requires a positive fitted conductance before interpreting its reciprocal as resistance.

The four training pairs are $(20, 1.1)$, $(40, 1.9)$, $(60, 3.1)$ and $(80, 3.9)$ in the stated units. They give $A = 12000$, $B = 596$, and $\widehat{G} = 596/12000 \simeq 0.049667$. Hence $\widehat{R} \simeq 20.134\,\mathrm{mmHg\,min/L}$. The precision here reflects arithmetic on a synthetic dataset; it is not a precision claim about a measured vascular resistance.

There is also a useful unit check. If the numerical pressure values are replaced by $x_i' = cx_i$ with $c > 0$, then $\widehat{G}' = \widehat{G}/c$. Consequently $\widehat{G}'x_i' = \widehat{G}x_i$: the physical predictions are unchanged. A unit conversion changes the coefficient's numerical value and units together.

3 A frozen rule on new observations

Slides for this section

The three held-out pairs are $(30, 1.4)$, $(70, 3.6)$ and $(90, 4.4)$. Using the already selected conductance, their predictions are 1.4900, 3.4767 and 4.4700 L/min. Their signed residuals are approximately 0.0900, -0.1233 and 0.0700 L/min. The mean absolute error is therefore

$$\mathrm{MAE} = \frac{0.0900 + 0.1233 + 0.0700}{3} \simeq 0.09444\,\mathrm{L/min}. \tag{3.1}$$

Absolute error retains the unit of flow. Displaying the three residuals preserves information that their average loses, such as which pressure range has the largest error.

For a simple comparison, a predictor that always returns the training mean flow returns 2.5 L/min. Its absolute errors on these same held-out points are 1.1, 1.1 and 1.9, averaging 1.3667 L/min. This comparison is additional arithmetic on the displayed observations, not a separate physiological model or a claim about a population. Three test observations provide a small check of prediction, not a reliable estimate of every future error.

If the held-out errors are used to choose a new offset or a new family of curves, the rows have become part of model development. Testing the revised rule requires other observations. The appropriate separation also depends on the intended claim: repeated measurements from one preparation do not automatically test performance on a new preparation.

4 Training coordinates and class centers

Slides for this section

The classification experiment has eighty synthetic cases with two features: a rate and a magnitude describing interval variation. Labels 0 and 1 identify two generator regimes, not diagnoses. Forty-eight cases define the model and thirty-two are held out. The generator clips both raw features below at a positive value, so these are explicitly constructed measurements rather than an unmodified Gaussian physiological model.

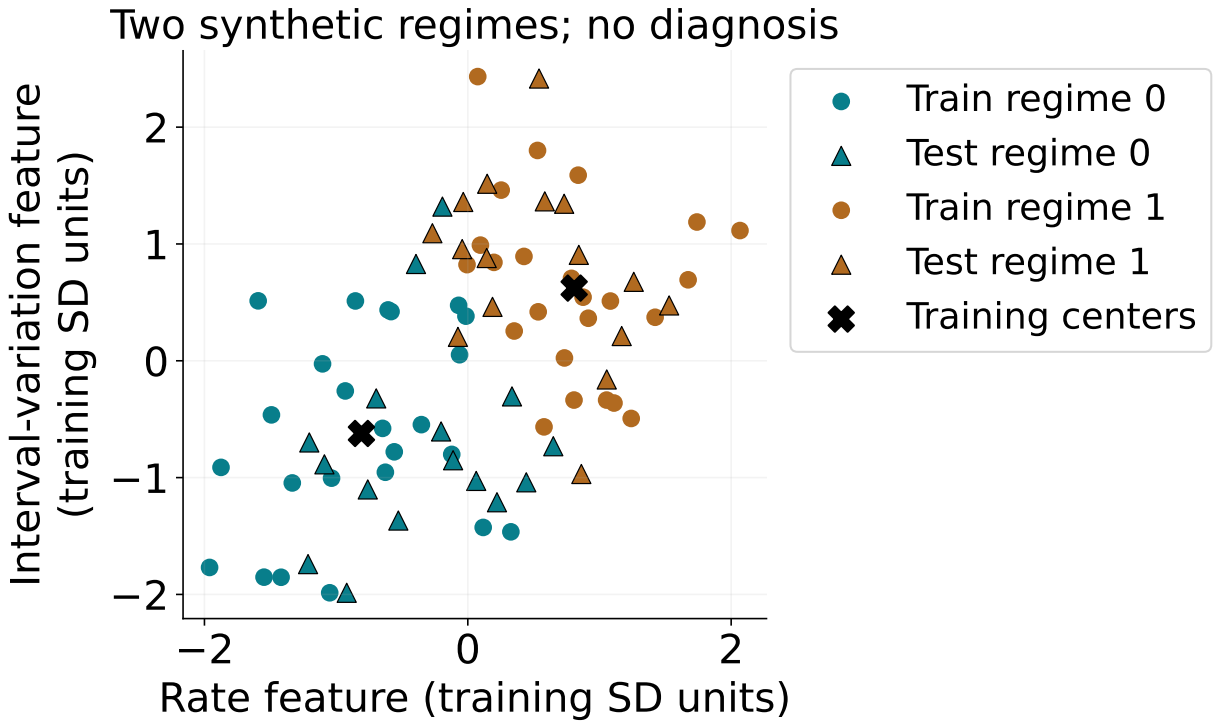
For feature j, compute a training mean μ_j and training standard deviation $s_j > 0$, then use

$$z_j = \frac{x_j - \mu_j}{s_j}. \tag{4.1}$$

The same μ_j, s_j transform every future row. A standardized value of one means one training standard deviation above the training mean. It does not give the two physical measurements the same biological interpretation. For label k, average its standardized training rows to obtain a center c_k. Classify a new point by its smaller squared distance:

$$\widehat{k}(z) = \operatorname*{argmin}_{k \in \{0,1\}} \|z - c_k\|^2, \qquad \|z - c_k\|^2 = \sum_{j=1}^{2} (z_j - c_{kj})^2. \tag{4.2}$$

The implementation assigns exact ties to label zero. Both feature scaling and the centers are fitted quantities. Computing the scale from all eighty rows would let the held-out cases alter the geometry used to classify themselves.



Training means and scales define the axes; circles are training cases, triangles are held-out cases, and crosses are training centers. Generator regimes have no diagnostic interpretation.

5 The boundary hidden in the distance comparison

Slides for this section

The nearest-center rule has a simple geometry. Label one is chosen when $\|z-c_1\|^2 < \|z-c_0\|^2$. Expanding both sides cancels $z_1^2 + z_2^2$ and gives

$$2z \cdot (c_1 - c_0) > \|c_1\|^2 - \|c_0\|^2. \tag{5.1}$$

Equality describes a straight line in the two-dimensional standardized coordinates, provided the centers differ. Thus the apparently nonlinear squared-distance comparison defines a linear boundary. It cannot follow an arbitrary curved separation between classes.

Scaling changes that geometry in raw coordinates. If $\overline{x}_{kj}$ is the raw class mean, then

$$\|z - c_k\|^2 = \sum_{j=1}^{2} \frac{(x_j - \overline{x}_{kj})^2}{s_j^2}.$$

A feature with larger training variation receives smaller weight per squared raw-unit difference. This explains why changing units consistently need not change the classification, whereas deciding on a different scaling rule can. The decision also discards the distinction between points near and far from both centers: nearest does not necessarily mean close. No calibrated probability is produced by this rule.

6 Count the errors the claim actually concerns

Slides for this section

With true labels as rows and predictions as columns, the held-out confusion table is

	$\widehat{k}=0$	$\widehat{k}=1$
$k=0$	12	4
$k=1$	0	16

There are 28 correct predictions out of 32, giving accuracy 0.875. Of the 16 true label-zero cases, 4 are mislabeled one. Of the 16 true label-one cases, none is mislabeled zero. The balanced test sample would give accuracy 0.5 to either constant-label rule. These counts concern generator labels, so interpreting them as clinical sensitivity or specificity would change the meaning of the target.

For a claim about new people, splitting individual windows is inadequate when windows from the same person can enter both sets. The grouping unit must be the person; training transformations and centers must then use training people only. Even that split does not repair inaccurate labels or a population unlike the intended application. The mathematical rule is completely specified here, but the scientific claim still depends on how observations and labels were obtained.

Exercises

MC08-E1. A small calibration

Synthetic observations are (10 mmHg, 0.6 L/min) and (30 mmHg, 1.8 L/min).
a Compare squared-error totals for G=0.05 and 0.06 L/(min mmHg).

b Find resistance for the better candidate, with units.

c Explain why perfect fit is not validation.

MC08-E2. Test the frozen rule

Freeze G=.04 L/(min mmHg). Held-out observations are (25 mmHg, 1.2 L/min) and (50 mmHg, 1.7 L/min).
a Calculate predictions, signed residuals predicted minus observed, and MAE.

b A colleague changes G after seeing these errors. What status do these rows now have, and how should the revised rule be evaluated?

c Name a useful additional test.

MC08-E3. Read a confusion table

For 20 new synthetic cases: true 0/pred 0=7; true 0/pred 1=3; true 1/pred 0=2; true 1/pred 1=8.
a Compute accuracy and both error counts.

b Compute the fraction of true 1 cases detected as 1.

c Why does that fraction not establish clinical sensitivity?

MC08-E4. A project evaluation plan

A project records five consecutive windows from each of ten volunteers. No data collection is required for this exercise.

a For a claim about new people, explain why randomly splitting windows is weak.

b Give a better split and specify when scaling is fitted.

c State one limitation that remains and explain how it affects the conclusion.

References

Zan Ahmad, Mathematical Cardiology syllabus, Fall 2023. Public PDF pages 1–2, Course Topics and Course Objectives.
https://zanahmad.com/files/Fall_2023_HEART_Syllabus.pdf

Zan Ahmad, original editable Mathematical Cardiology syllabus. Rendered export pages 2–3, Assignments & Grading and Projects.

Zan Ahmad, Introduction to Scientific Python, Lecture 5; source lineage includes Phillip Kerger (2023). PDF pages 5–12, learning tasks and linear regression.

scikit-learn, Common pitfalls and recommended practices. Sections 12.1 and 12.2, inconsistent preprocessing and data leakage; read 2026-11-14.
https://scikit-learn.org/stable/common_pitfalls.html