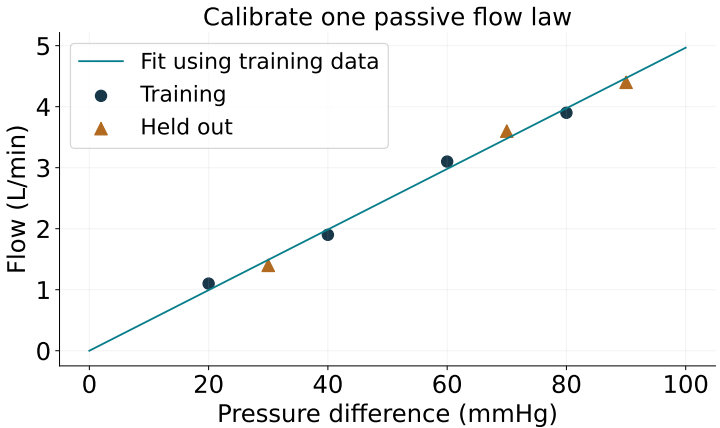
Mathematical Cardiology Lecture 8

Calibrating a flow law and constructing a classifier

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Fall 2023

A parameter inside a physical relation



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G has units $\mathrm{L}/(\min \mathrm{mmHg})$. A line through the origin asserts zero passive flow at equal boundary pressures.

Define the discrepancy

For training pairs (x_i, y_i),

$$r_i(G) = Gx_i - y_i, \qquad S(G) = \sum_i r_i(G)^2.$$

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Opposite signed errors do not cancel after squaring. The objective has units $(\mathrm{L/min})^2$.

Expand the objective

$$S(G) = \sum_i (G^2 x_i^2 - 2G x_i y_i + y_i^2).$$

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Write

$$S(G) = AG^2 - 2BG + C,$$

where $A = \sum x_i^2$, $B = \sum x_i y_i$, $C = \sum y_i^2$.

Complete one square

When $A > 0$,

$$S(G) = A(G - B/A)^2 + C - B^2/A.$$

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The square is smallest at zero, so

$$\boxed{\widehat{G} = B/A}.$$

Evaluate the fitted conductance

For the four displayed training observations,

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$$A = 12000, \qquad B = 596.$$

$$\widehat{G} = 0.049667, \qquad \widehat{R} = 20.134\,\mathrm{mmHg\,min/L}.$$

These digits describe this synthetic sample.

An experiment with no slope information

If every training pressure difference is zero,

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Every G gives the same objective. More zero-pressure observations do not identify conductance.

A unit conversion check

Let $x_i' = cx_i$ with $c > 0$.

$$\widehat{G}' = \frac{c \sum x_i y_i}{c^2 \sum x_i^2} = \widehat{G}/c.$$

A unit conversion check

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$$\widehat{G}' = \frac{c \sum x_i y_i}{c^2 \sum x_i^2} = \widehat{G}/c.$$

$$\widehat{G}' x_i' = \widehat{G} x_i.$$

The prediction stays the same.

Evaluate the frozen line

x	30	70	90
y	1.4	3.6	4.4
$\widehat{y}$	1.4900	3.4767	4.4700
r	0.0900	-0.1233	0.0700

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$\mathrm{MAE} \simeq 0.09444\,\mathrm{L/min}.$

Compare with a constant predictor

The training mean is $2.5\,\mathrm{L/min}$.

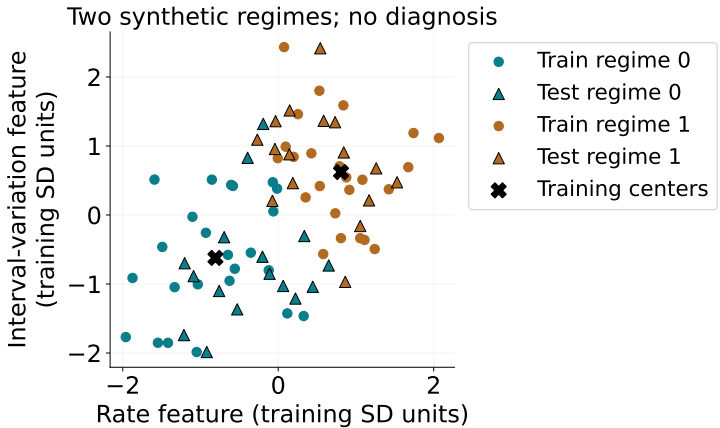
Compare with a constant predictor

The training mean is $2.5\,\mathrm{L/min}$. Its held-out absolute errors are 1.1, 1.1, 1.9.

$$\mathrm{MAE}_{\mathrm{constant}} = 4.1/3 \simeq 1.3667\,\mathrm{L/min}.$$

Both rules are evaluated on the same three cases.

Fit a coordinate system on training cases



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Compute μ_j, s_j from the 48 training cases. Use those same values for the 32 held-out cases.

The nearer training center

Let c_0, c_1 be the class means in standardized coordinates.

$$d_k(z) = \sum_{j=1}^{2} (z_j - c_{kj})^2.$$

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Choose the smaller d_k; exact ties go to zero in this implementation.

Expand the decision boundary

Label one requires

$$\|z - c_1\|^2 < \|z - c_0\|^2.$$

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The $\|z\|^2$ terms cancel:

$$2z \cdot (c_1 - c_0) > \|c_1\|^2 - \|c_0\|^2.$$

Equality is a line when the centers differ.

Scaling determines the metric

In raw coordinates,

$$d_k(x) = \sum_{j=1}^{2} \frac{(x_j - \overline{x}_{kj})^2}{s_j^2}.$$

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The training variations determine the weights. A new test mean would change the fitted rule.

Read both directions of error

true\predicted	0	1
0	12	4
1	0	16

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Accuracy is $28/32 = 0.875$. All four errors are $0 \to 1$.

The held-out unit belongs to the claim

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A claim about new people requires whole people to be held out. Windows from one person can share information even if the windows differ.
Choose the grouping first; fit scaling and the rule on training groups.

After inspecting an error

Changing the model in response to test errors makes those cases development data.

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Changing the model in response to test errors makes those cases development data.
Retain the original result. Evaluate the changed rule on other observations.
A transparent fit is useful because its assumptions and failures can be named.