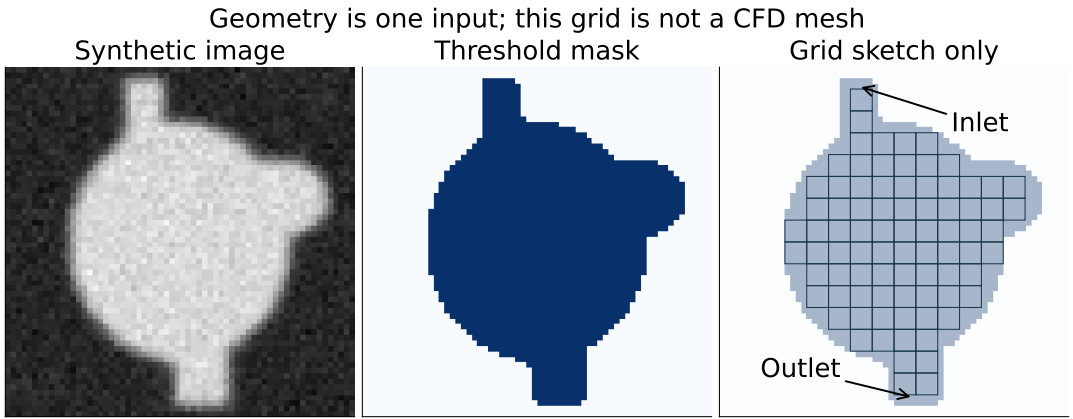
Mathematical Cardiology Lecture 9

# From image geometry to a tracer balance

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# A picture contains intensities



# A picture contains intensities

$$I = (I_{ij}), \qquad M_{ij}(\theta) = \mathbf{1}\{I_{ij} \geq \theta\}.$$

The image, the mask and the computational region are different objects.

# Increasing a threshold selects a subset

If $\theta_2 > \theta_1$,

$$I_{ij} \geq \theta_2 \Longrightarrow I_{ij} \geq \theta_1.$$

Thus $N(\theta_2) \leq N(\theta_1)$.

$$1699 \geq 1641 \geq 1563 \quad \text{at } 0.4, 0.5, 0.6.$$

# Pixel count needs a physical scale

For spacings $s_x, s_y$,

$$A = N s_x s_y.$$

Changing a pixel count does not determine a volume. Out-of-plane geometry is missing from a single two-dimensional mask.

# A small area change can alter a route

A narrow neck may contain few pixels. Removing those pixels can disconnect the pocket while changing total area only slightly.
Inspect the neck, top inlet, lower outlet and remaining wall boundary.

# Conservation on each cell

With outward volume flux positive,

$$\sum_{f \in \partial \text{cell}} Q_f = 0.$$

At a shared face, one cell's outward flow is the other's inward flow. Internal fluxes cancel when cells are added.

# The spatial problem needs more information

The top opening needs an inlet condition; the lower opening needs an outlet condition. Walls, fluid properties and momentum balance determine how a velocity field can develop. A mesh supplies locations for the calculation, not these laws.

# Reduce the region to one mixed volume

$$M = VC.$$

$M$ is tracer amount and $C$ is tracer amount per fluid volume. Equal inflow and outflow $Q$ keep $V$ fixed. Complete mixing makes outlet concentration equal to $C$.

# Balance the tracer over a short interval

$$M(t+h) - M(t) \simeq hQC_{\text{in}} - hQC(t).$$

For tracer-free inflow,

$$C_{n+1} = (1 - hQ/V)C_n.$$

# A time-step condition

For $C_n \geq 0$, the explicit update stays nonnegative if

$$0 \leq hQ/V \leq 1.$$

A larger step can create a negative numerical concentration. The companion uses the exact exponential, not this update.

# Repeated removal gives an exponential

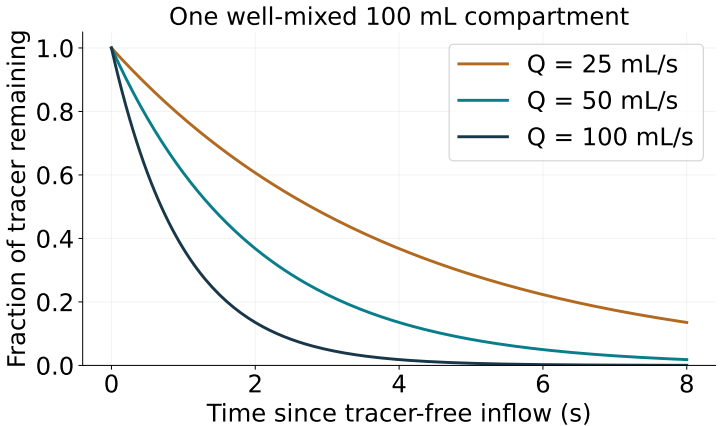
Use $n$ steps over time $t$:

$$C_n/C_0 = (1 - Qt/(nV))^n.$$

As the steps become finer,

$$\boxed{C(t)/C_0 = e^{-Qt/V}}.$$

# The time constant is a volume-to-flow ratio



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$$\tau = V/Q, \qquad C(t)/C_0 = e^{-t/\tau}.$$

At $t = \tau$, the remaining fraction is $e^{-1} \simeq 0.368$. At half-time, $t_{1/2} = \tau \log 2$.

# Evaluate the three curves

At $V = 100\,\mathrm{mL}$ and $t = 2\,\mathrm{s}$,

| $Q$ (mL/s) | 25 | 50 | 100 |
|---|---|---|---|
| $\tau$ (s) | 4 | 2 | 1 |
| $C/C_0$ | 0.607 | 0.368 | 0.135 |

# An identifiability limitation

For any $a > 0$,

$$\frac{aQ}{aV} = \frac{Q}{V}.$$

The decay curve cannot distinguish $(V, Q)$ from $(aV, aQ)$. If $Q = 0$, concentration remains constant.

# A pocket needs its own state

An optional two-compartment extension is

$$V_a \dot{C}_a = -Q C_a + q(C_b - C_a),$$

$$V_b \dot{C}_b = q(C_a - C_b).$$

The companion does not solve these equations.

# Add the two balances

$$\frac{\mathrm{d}}{\mathrm{d}t}(V_a C_a + V_b C_b) = -Q C_a.$$

Internal exchange cancels. If $q = 0$, tracer initially in the pocket remains there. One mixed concentration cannot represent this case.

# Two different tests

Keep physics fixed and refine mesh and time step: compare a named output for numerical sensitivity. Compare that output with independent measurements: test the physical model.
Neither comparison alone turns washout into a stroke prediction.