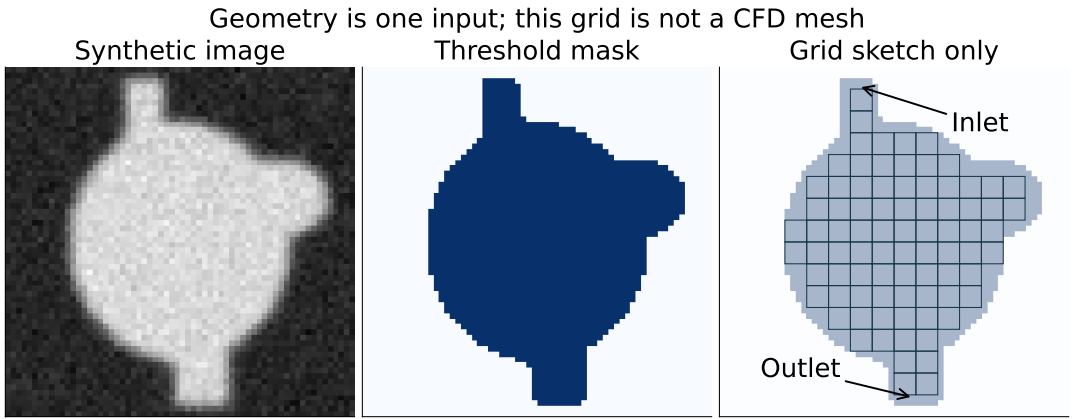
Mathematical Cardiology Lecture 9

From image geometry to a tracer balance

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A picture contains intensities



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$$I = (I_{ij}), \qquad M_{ij}(\theta) = \mathbf{1}\{I_{ij} \geq \theta\}.$$

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The image, the mask and the computational region are different objects.

Increasing a threshold selects a subset

If $\theta_2 > \theta_1$,

$$I_{ij} \geq \theta_2 \implies I_{ij} \geq \theta_1.$$

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If $\theta_2 > \theta_1$,

$$I_{ij} \geq \theta_2 \Longrightarrow I_{ij} \geq \theta_1.$$

Thus $N(\theta_2) \leq N(\theta_1)$.

$$1699 \geq 1641 \geq 1563 \quad \text{at } 0.4, 0.5, 0.6.$$

Pixel count needs a physical scale

For spacings s_x, s_y,

$$A = N s_x s_y.$$

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Changing a pixel count does not determine a volume. Out-of-plane geometry is missing from a single two-dimensional mask.

A small area change can alter a route

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A narrow neck may contain few pixels. Removing those pixels can disconnect the pocket while changing total area only slightly.
Inspect the neck, top inlet, lower outlet and remaining wall boundary.

Conservation on each cell

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$$\sum_{f \in \partial \text{cell}} Q_f = 0.$$

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At a shared face, one cell's outward flow is the other's inward flow. Internal fluxes cancel when cells are added.

The spatial problem needs more information

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The top opening needs an inlet condition; the lower opening needs an outlet condition. Walls, fluid properties and momentum balance determine how a velocity field can develop. A mesh supplies locations for the calculation, not these laws.

Reduce the region to one mixed volume

$$M = VC.$$

M is tracer amount and C is tracer amount per fluid volume.

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Balance the tracer over a short interval

$$M(t+h) - M(t) \simeq hQC_{\text{in}} - hQC(t).$$

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For tracer-free inflow,

$$C_{n+1} = (1 - hQ/V)C_n.$$

A time-step condition

For $C_n \geq 0$, the explicit update stays nonnegative if

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A larger step can create a negative numerical concentration. The companion uses the exact exponential, not this update.

Repeated removal gives an exponential

Use n steps over time t:

$$C_n/C_0 = (1 - Qt/(nV))^n.$$

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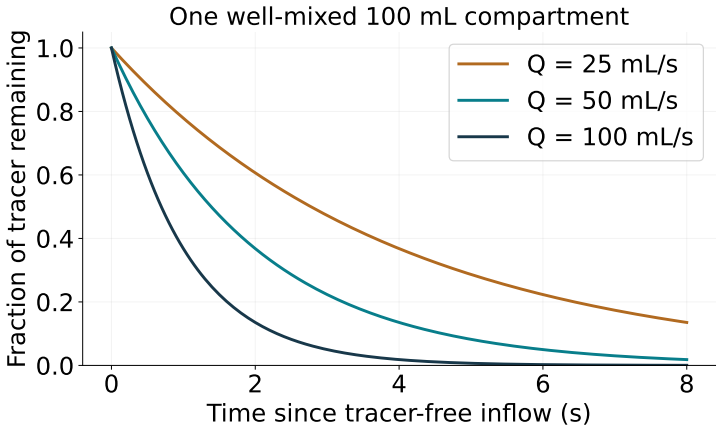
Use n steps over time t:

$$C_n/C_0 = (1 - Qt/(nV))^n.$$

As the steps become finer,

$$\boxed{C(t)/C_0 = e^{-Qt/V}}.$$

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At $t = \tau$, the remaining fraction is $e^{-1} \simeq 0.368$. At half-time, $t_{1/2} = \tau \log 2$.

Evaluate the three curves

At $V = 100\,\mathrm{mL}$ and $t = 2\,\mathrm{s}$,

Q (mL/s)	25	50	100
τ (s)	4	2	1
C/C_0	0.607	0.368	0.135

An identifiability limitation

For any $a > 0$,

$$\frac{aQ}{aV} = \frac{Q}{V}.$$

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The decay curve cannot distinguish (V, Q) from (aV, aQ). If $Q = 0$, concentration remains constant.

A pocket needs its own state

An optional two-compartment extension is

$$V_a \dot{C}_a = -Q C_a + q(C_b - C_a),$$

$$V_b \dot{C}_b = q(C_a - C_b).$$

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The companion does not solve these equations.

Add the two balances

$$\frac{\mathrm{d}}{\mathrm{d}t}(V_a C_a + V_b C_b) = -Q C_a.$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}(V_a C_a + V_b C_b) = -Q C_a.$$

Internal exchange cancels. If $q = 0$, tracer initially in the pocket remains there. One mixed concentration cannot represent this case.

Two different tests

Keep physics fixed and refine mesh and time step: compare a named output for numerical sensitivity.

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Keep physics fixed and refine mesh and time step: compare a named output for numerical sensitivity. Compare that output with independent measurements: test the physical model.
Neither comparison alone turns washout into a stroke prediction.