Resistance networks: constitutive laws, pressure, and conservation

Lecture notes

Conservation constrains how flows fit together but does not determine them. A constitutive relation supplies the missing connection between pressure difference and flow. We combine linear resistance laws with nodal conservation, derive the resulting matrix problem, and solve a five-link example. Signs, boundary conditions, and a positivity argument explain why the answer is determined. These arguments also produce checks that do not merely repeat the linear solve.

1 Linear resistance is a local constitutive assumption

Slides for this section

For an open link joining nodes i and j, choose positive flow from i to j and write

$$Q_{i\to j} = G_{ij}(P_i - P_j), \qquad G_{ij} = R_{ij}^{-1} > 0.$$

Pressure is measured in mmHg, flow in L/min, resistance in mmHg min/L, and conductance in L/(min mmHg). A negative $Q_{i\to j}$ means flow opposite the reference arrow. Reversing the arrow reverses the signed variable and its pressure difference, while preserving the physical flow.

The law represents a dissipative link with a fixed linear response. It does not represent a valve: an open resistor allows either direction. It also excludes storage within the link and ignores inertial transients, diameter changes, collapse, and pressure-dependent material response. The volume-storage relation $V = V_d + CP$ is a different constitutive law. A resistor determines a flow from a pressure difference; a compliance determines stored volume from a distending pressure.

The link dissipates mechanical power proportional to

$$(P_i - P_j)Q_{i\to j} = G_{ij}(P_i - P_j)^2 = R_{ij}Q_{i\to j}^2 \geq 0.$$

The expression has pressure times volume per time units; a conversion is required to report watts. Negative signed flow does not imply negative dissipation.

2 Boundary pressures are data; interior pressures are unknowns

Slides for this section

Use two prescribed endpoint pressures, $P_H = 96$ and $P_L = 6$ mmHg. The interior nodes are A and B. Their links have resistances

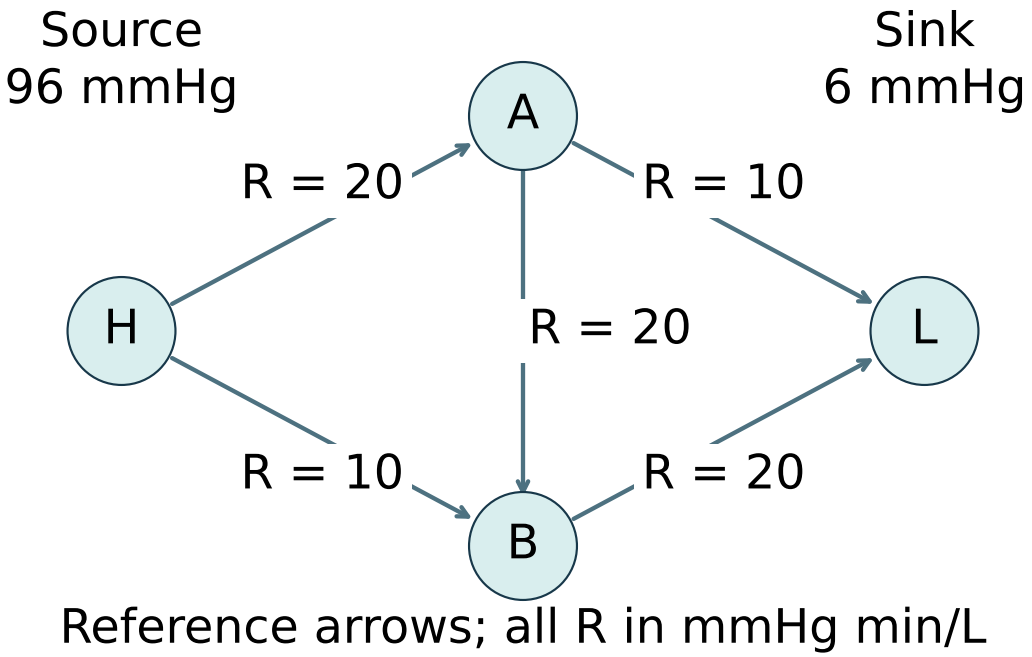
$$R_{HA} = 20, \quad R_{AL} = 10, \quad R_{HB} = 10, \quad R_{BL} = 20, \quad R_{AB} = 20$$

in mmHg min/L. The bridge arrow is $A \to B$. Prescribed endpoints may be maintained by external pumps or reservoirs; they are not unknowns solved by conservation at A and B.

At a non-storing interior node, total inflow equals total outflow. Combining that statement with the constitutive law gives

$$0 = \frac{96 - P_A}{20} - \frac{P_A - 6}{10} - \frac{P_A - P_B}{20},$$
$$0 = \frac{96 - P_B}{10} + \frac{P_A - P_B}{20} - \frac{P_B - 6}{20}.$$

The bridge occurs with opposite signs. These equations would instead have $\dot{V}_A$ and $\dot{V}_B$ on the left in a model with storage. Steady flow is an approximation imposed here, not a general property of vessels.



Original synthetic network. Endpoint pressures are fixed; internal pressures follow from conservation. Arrows declare reference directions, and every link is an open resistor.

3 Solve the two unknowns, then recover every flow

Slides for this section

Multiply both equations by the resistance unit value 20 and collect unknown pressures:

$$4P_A - P_B = 108, \qquad -P_A + 4P_B = 198.$$

The coefficients arise from the conductances incident on each node. Eliminating P_B gives $15P_A = 630$, hence

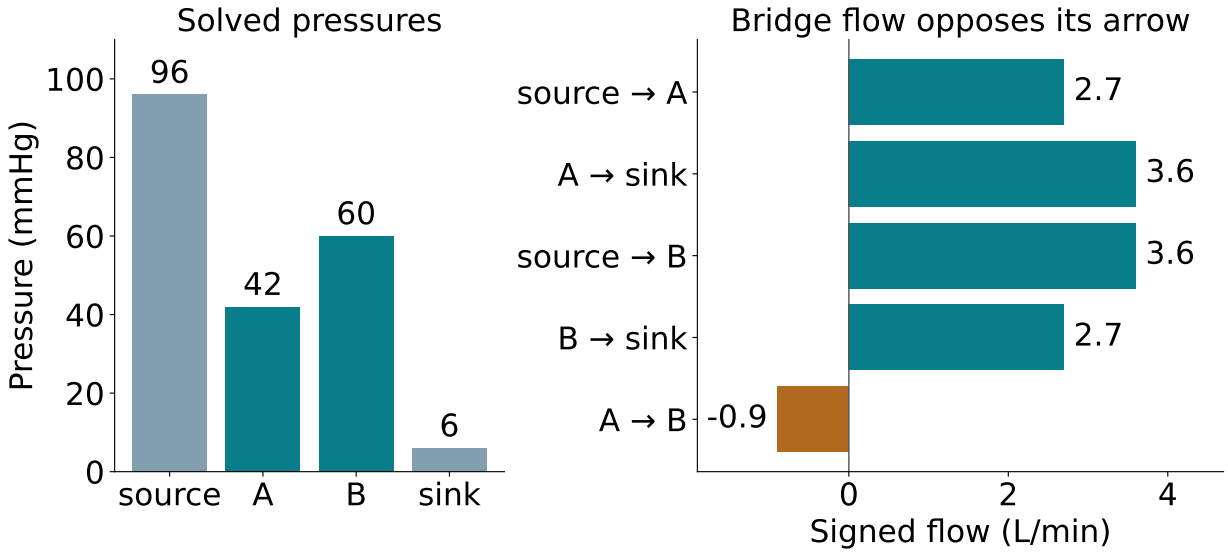
$$\boxed{P_A = 42 \text{ mmHg}, \qquad P_B = 60 \text{ mmHg}.}$$

Substitution into the five local laws gives

reference link	$H \to A$	$A \to L$	$H \to B$	$B \to L$	$A \to B$
Q (L/min)	2.7	3.6	3.6	2.7	−0.9

The actual bridge flow is 0.9 L/min from B to A. Audit A: $2.7 - 3.6 - (-0.9) = 0$. Audit B: $3.6 + (-0.9) - 2.7 = 0$. The external inlet and outlet totals both equal 6.3 L/min.

The effective resistance between the prescribed endpoints is $(96-6)/6.3 = 100/7 \approx 14.286$ mmHg min/L. It is a property of this complete network, not the resistance of a single branch.



The solved internal pressures are 42 and 60 mmHg. Bridge flow is -0.9 L/min in the A-to-B reference, so its actual direction is B to A.

4 The conductance matrix and a uniqueness argument

Slides for this section

For an arbitrary interior node i, conservation is

$$\sum_{j \sim i} G_{ij}(P_i - P_j) = 0.$$

Move known boundary pressures to the right. If $\mathbf{p}$ contains the unknown interior pressures, the system is $L_{II}\mathbf{p} = \mathbf{b}$, where diagonal entries sum all incident conductances and off-diagonal entries equal $-G_{ij}$. The example is

$$\begin{bmatrix} .20 & -.05 \\ -.05 & .20 \end{bmatrix} \begin{bmatrix} P_A \\ P_B \end{bmatrix} = \begin{bmatrix} 5.4 \\ 9.9 \end{bmatrix}.$$

The right-hand side has flow units. To see uniqueness, let $\mathbf{z}$ be the difference of two solutions with identical boundary values. Extend z by zero at boundary nodes. Then

$$\mathbf{z}^T L_{II} \mathbf{z} = \sum_{\{i,j\}} G_{ij}(z_i - z_j)^2.$$

Each undirected edge is counted once. The sum can vanish only if z is constant along every connected path. If every interior connected component reaches a prescribed-pressure node, that constant must be zero. Thus L_{II} is positive definite and the solution is unique. An isolated component with no pressure reference instead has an undetermined additive constant.

A second consequence is the weighted-average identity

$$P_i = \frac{\sum_{j \sim i} G_{ij} P_j}{\sum_{j \sim i} G_{ij}}.$$

An interior pressure therefore lies between the minimum and maximum prescribed boundary pressures, under the same connectivity assumption. A result outside that range signals an error or an additional source not present in this passive model.

5 Transformations that expose errors

Slides for this section

Adding one constant c to every prescribed and computed pressure leaves every difference, and hence every flow, unchanged. The zero of pressure is a reference choice in this resistance-only model. This does not authorize shifting internal pressure without shifting the external pressure in a compliance law: distending pressure is itself a difference.

If every resistance is multiplied by $k > 0$, every conductance is divided by k. Both sides of the interior matrix problem scale by $1/k$, so the interior pressures remain unchanged while every flow is divided by k. Doubling only one resistance is a different experiment and redistributes the pressure field.

For two links in series with no branch, conservation gives one common flow. Adding pressure drops gives $R_{\text{eq}} = R_1 + R_2$. For links in parallel between the same endpoints, adding flows gives $G_{\text{eq}} = G_1 + G_2$. These reductions apply only when the relevant topology really is series or parallel. The bridge in the five-link example prevents reducing its four outer links as two independent series paths.

6 Computational residuals and the scope of a claim

Slides for this section

Compute pressures by a linear solve rather than forming an explicit matrix inverse. Recover each edge flow once, use that same value in both endpoint accounts, and report the residual $\mathbf{r} = L_{II}\mathbf{p} - \mathbf{b}$. A small residual is evidence that the assembled equations were solved, not that the correct network was assembled.

Complement it with the hand-computed five flows, equal external inlet and outlet totals, reference-pressure invariance, resistance scaling, and the interior pressure bounds. Check positive finite resistances and finite assembled coefficients; an overflow is a failed calculation, not a new physiological regime.

To test a real network, compare predicted flows with independent measurements at specified pressure boundaries. A mismatch can arise from a wrong resistance, an omitted branch, storage during a transient, or a nonlinear vessel response. The present model predicts neither pulse-wave propagation nor the contraction of the heart. It isolates a question that can be answered completely: how do prescribed endpoint pressures distribute steady flow through a passive resistive graph?

Exercises

M2-1. Interpret the arrow

An open resistor joins A at 35 mmHg to B at 50 mmHg. R=5 mmHg min/L; positive flow is defined from A toward B.
a Calculate the signed reference flow and its physical direction.

b Redraw the reference arrow from B toward A. What changes?

c Would setting the negative value to 0 preserve the stated resistor law?

M2-2. Two resistors in series

A source at 80 mmHg connects through R1=10 mmHg min/L to node X, then through R2=5 mmHg min/L to a sink at 5 mmHg. X has no other connection and no accumulation.
a Explain why the two flows must be equal and calculate their value.

b Find the pressure at X and check the downstream link.

c If both resistances double with boundary pressures fixed, what happens?

M2-3. Audit a branch report

A steady junction receives 7 L/min from one link and sends 2 L/min through a second. A third reference arrow points outward from the junction.
a Find the third signed flow required by balance.

b A report instead lists Q3=6 L/min. What accumulation rate would those three flows imply?

c Give two distinct possible causes of the inconsistent report and an observation that could distinguish them.

M2-4. Use an informative limiting case

Keep the original five positive resistances but set both boundary pressures to 20 mmHg.
a Propose the internal pressures and all five flows.

b Verify why the node balances hold.

c A modified program reports nonzero circulating flow. Name one implementation check and one model change that could explain it.

References

Zan Ahmad, Fall 2024 HEART syllabus. PDF pages 1–2.

Hoppensteadt and Peskin, Modeling and Simulation in Medicine and the Life Sciences, second edition. Printed pages 8–10; PDF pages 21–23 in the supplied copy.