Mathematical Modeling Lecture 2

Resistance networks: constitutive laws, pressure, and conservation

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Fall 2024

Conservation needs a constitutive law

At a non-storing node, inflow equals outflow.

$$Q_{i\to j} = \frac{P_i - P_j}{R_{ij}} = G_{ij}(P_i - P_j).$$

The first statement connects links; the second describes one link.

Choose an orientation once

$$Q_{A\to B} = G(P_A - P_B), \qquad Q_{B\to A} = -Q_{A\to B}.$$

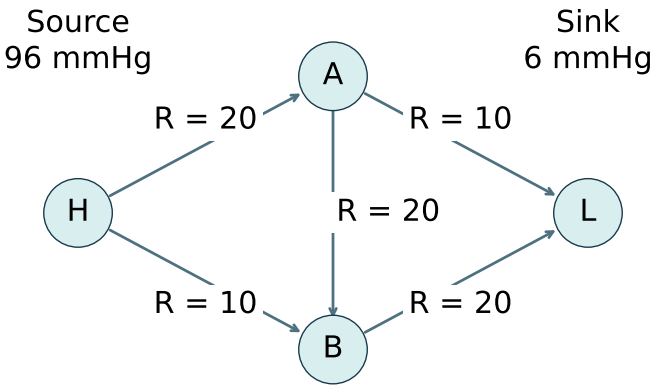
If $P_B > P_A$, the first variable is negative. A reference arrow is not an ideal valve.

Dimensions and dissipation

$$[R] = \frac{\text{mmHg}\,\text{min}}{\text{L}}, \qquad [G] = \frac{\text{L}}{\text{min}\,\text{mmHg}}.$$

$$(P_i - P_j)Q_{i\to j} = R_{ij}Q_{i\to j}^2 \geq 0.$$

Prescribed endpoints and unknown interiors



Prescribed endpoints and unknown interiors

$P_H = 96$, $P_L = 6$ mmHg; solve for P_A, P_B.

$$R_{HA} = 20,\ R_{AL} = 10,\ R_{HB} = 10,\ R_{BL} = 20,\ R_{AB} = 20.$$

Every link is open and has positive resistance.

Write the A balance

$$\underbrace{\frac{96 - P_A}{20}}_{H \to A} - \underbrace{\frac{P_A - 6}{10}}_{A \to L} - \underbrace{\frac{P_A - P_B}{20}}_{A \to B} = 0.$$

Each term is a flow in L/min.

Write the B balance

$$\frac{96 - P_B}{10} + \frac{P_A - P_B}{20} - \frac{P_B - 6}{20} = 0.$$

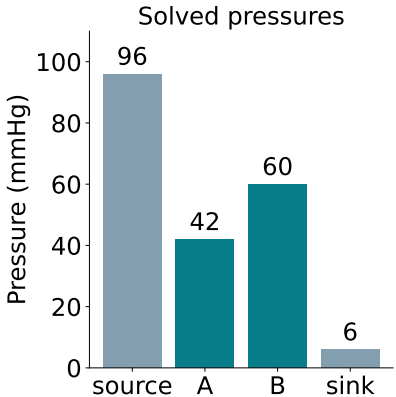
The bridge term has the opposite sign from its appearance at A.

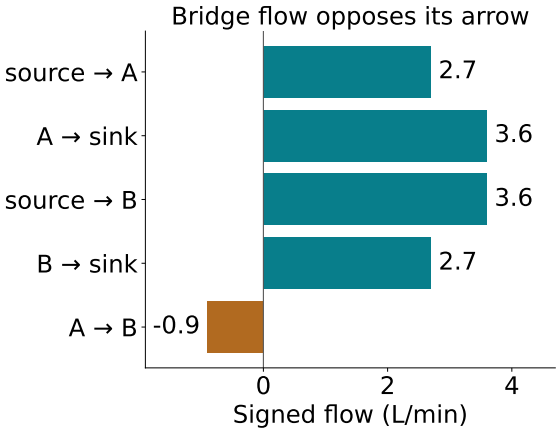
Collect and eliminate

$$4P_A - P_B = 108, \qquad -P_A + 4P_B = 198.$$

$$15P_A = 630 \quad \Rightarrow \quad P_A = 42,\ P_B = 60 \text{ mmHg}.$$

Recover five signed flows





Recover five signed flows

$$Q_{HA} = 2.7,\ Q_{AL} = 3.6,\ Q_{HB} = 3.6,\ Q_{BL} = 2.7,$$
$$Q_{AB} = (42 - 60)/20 = -0.9 \quad [\mathrm{L/min}].$$

The bridge carries fluid from B to A.

Independent node and boundary audits

$$A:\ 2.7 - 3.6 + 0.9 = 0, \qquad B:\ 3.6 - 0.9 - 2.7 = 0.$$

$$Q_{\text{in}} = 2.7 + 3.6 = 6.3 = 3.6 + 2.7 = Q_{\text{out}}.$$

The graph matrix

$$\sum_j G_{ij}(P_i - P_j) = 0 \quad \Longrightarrow \quad L_{II}\mathbf{p} = \mathbf{b}.$$

$$L_{II} = \begin{bmatrix} .20 & -.05 \\ -.05 & .20 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 5.4 \\ 9.9 \end{bmatrix}.$$

Why the solution is unique

Let $z = 0$ on prescribed-pressure nodes.

$$\mathbf{z}^T L_{II} \mathbf{z} = \sum_{\{i,j\}} G_{ij}(z_i - z_j)^2 \geq 0.$$

If each interior component reaches a boundary, equality implies $z = 0$.

An interior pressure is a weighted mean

$$P_i = \frac{\sum_j G_{ij} P_j}{\sum_j G_{ij}}.$$

Therefore $6 \leq P_i \leq 96$ mmHg in this connected passive example. An interior source would change this conclusion.

Change the pressure reference

$$P_i' = P_i + c \quad \Rightarrow \quad P_i' - P_j' = P_i - P_j.$$

All flows are unchanged. Shift every endpoint and interior pressure consistently.

Scale every resistance

$$R'_e = kR_e, \quad G'_e = G_e/k.$$

$$(L_{II}/k)\mathbf{p}' = \mathbf{b}/k \Rightarrow \mathbf{p}' = \mathbf{p}, \quad Q'_e = Q_e/k.$$

Series and parallel follow from balances

In series, one common Q gives $\Delta P = Q(R_1 + R_2)$.
In parallel, one common ΔP gives $Q_{\text{total}} = (G_1 + G_2)\Delta P$.

$$R_{\text{eff}} = \frac{90}{6.3} = \frac{100}{7}.$$

A numerical residual is necessary

$$\mathbf{r} = L_{II}\mathbf{p} - \mathbf{b}.$$

Solve the matrix system, then recover flows. Add reference-shift and resistance-scaling tests; do not use an inverse as an intermediate numerical object.

Dynamic extensions: storage, valves, and pumping

Storage introduces $\dot{V}_i$, a valve restricts direction, and active contraction changes a pressure–volume relation.
Each mechanism changes a specific equation. None is supplied by the present steady resistor solve.