Mathematical Modeling Lecture 2

Resistance networks: constitutive laws, pressure, and conservation

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Conservation needs a constitutive law

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$$Q_{i\to j} = \frac{P_i - P_j}{R_{ij}} = G_{ij}(P_i - P_j).$$

The first statement connects links; the second describes one link.

Choose an orientation once

$$Q_{A\to B} = G(P_A - P_B), \qquad Q_{B\to A} = -Q_{A\to B}.$$

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If $P_B > P_A$, the first variable is negative. A reference arrow is not an ideal valve.

Dimensions and dissipation

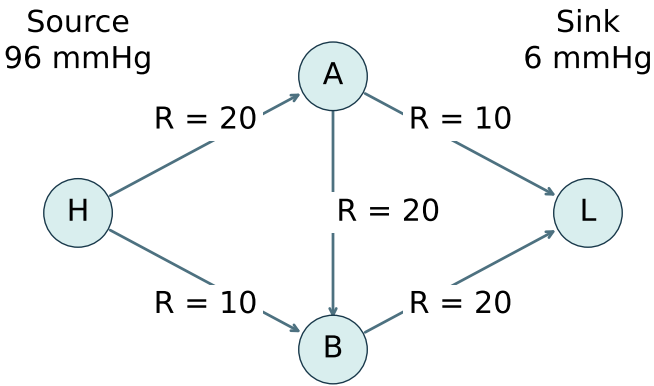
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$$(P_i - P_j)Q_{i\to j} = R_{ij}Q_{i\to j}^2 \geq 0.$$

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$$R_{HA} = 20,\ R_{AL} = 10,\ R_{HB} = 10,\ R_{BL} = 20,\ R_{AB} = 20.$$

Every link is open and has positive resistance.

Write the A balance

$$\underbrace{\frac{96 - P_A}{20}}_{H \to A} - \underbrace{\frac{P_A - 6}{10}}_{A \to L} - \underbrace{\frac{P_A - P_B}{20}}_{A \to B} = 0.$$

Write the A balance

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Each term is a flow in L/min.

Write the B balance

$$\frac{96 - P_B}{10} + \frac{P_A - P_B}{20} - \frac{P_B - 6}{20} = 0.$$

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The bridge term has the opposite sign from its appearance at A.

Collect and eliminate

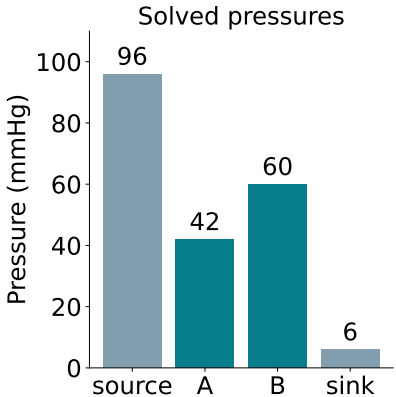
$$4P_A - P_B = 108, \qquad -P_A + 4P_B = 198.$$

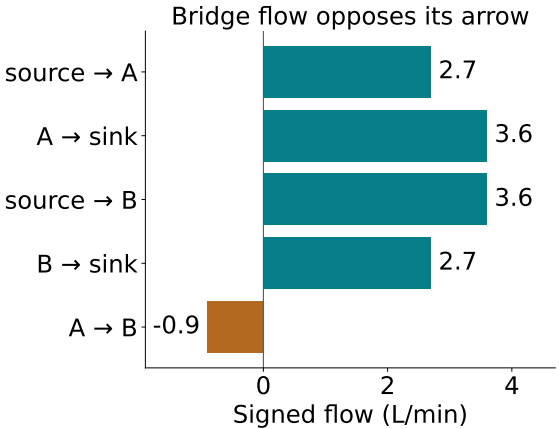
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$$4P_A - P_B = 108, \qquad -P_A + 4P_B = 198.$$

$$15P_A = 630 \quad \Rightarrow \quad P_A = 42,\ P_B = 60 \text{ mmHg}.$$

Recover five signed flows





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$$Q_{HA} = 2.7,\ Q_{AL} = 3.6,\ Q_{HB} = 3.6,\ Q_{BL} = 2.7,$$
$$Q_{AB} = (42 - 60)/20 = -0.9 \quad [\mathrm{L/min}].$$

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The bridge carries fluid from B to A.

Independent node and boundary audits

$$A:\ 2.7 - 3.6 + 0.9 = 0, \qquad B:\ 3.6 - 0.9 - 2.7 = 0.$$

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$$Q_{\text{in}} = 2.7 + 3.6 = 6.3 = 3.6 + 2.7 = Q_{\text{out}}.$$

The graph matrix

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$$L_{II} = \begin{bmatrix} .20 & -.05 \\ -.05 & .20 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 5.4 \\ 9.9 \end{bmatrix}.$$

Why the solution is unique

Let $z = 0$ on prescribed-pressure nodes.

$$\mathbf{z}^T L_{II} \mathbf{z} = \sum_{\{i,j\}} G_{ij}(z_i - z_j)^2 \geq 0.$$

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If each interior component reaches a boundary, equality implies $z = 0$.

An interior pressure is a weighted mean

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Therefore $6 \leq P_i \leq 96$ mmHg in this connected passive example. An interior source would change this conclusion.

Change the pressure reference

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All flows are unchanged. Shift every endpoint and interior pressure consistently.

Scale every resistance

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$$(L_{II}/k)\mathbf{p}' = \mathbf{b}/k \Rightarrow \mathbf{p}' = \mathbf{p}, \quad Q'_e = Q_e/k.$$

Series and parallel follow from balances

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$$R_{\text{eff}} = \frac{90}{6.3} = \frac{100}{7}.$$

A numerical residual is necessary

$$\mathbf{r} = L_{II}\mathbf{p} - \mathbf{b}.$$

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$$\mathbf{r} = L_{II}\mathbf{p} - \mathbf{b}.$$

Solve the matrix system, then recover flows. Add reference-shift and resistance-scaling tests; do not use an inverse as an intermediate numerical object.

Dynamic extensions: storage, valves, and pumping

Storage introduces $\dot{V}_i$, a valve restricts direction, and active contraction changes a pressure–volume relation.

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Each mechanism changes a specific equation. None is supplied by the present steady resistor solve.