Mathematical Modeling Lecture 3

A mixed chamber: differential equations and numerical stability

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Fall 2024

Mixing and mass outflow

$$c = M/W, \qquad Qc = (Q/W)M.$$

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Define $k = Q/W$ and mass input r.

$$\dot{M} = r - kM.$$

Equilibrium and direction of motion

$$M_* = r/k, \qquad \dot{M} = k(M_* - M).$$

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For $r = 2$, $k = .25$, the equilibrium is 8 mg. Below it the mass rises; above it the mass falls.

Integrating-factor derivation

Let $e = M - M_*$. Then $\dot{e} = -ke$.

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$$\frac{d}{dt}(e^{kt}e) = 0 \quad \Rightarrow \quad e(t) = e(0)e^{-kt}.$$

Convex-combination form

$$M(t) = e^{-kt}M_0 + (1 - e^{-kt})M_*.$$

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Nonnegative weights summing to one imply positivity and confinement between the two endpoints.

Relaxation time and dimensionless equation

$$\tau = 1/k, \quad \theta = kt, \quad m = M/M_* \quad (r > 0).$$

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$$dm/d\theta = 1 - m.$$

After one τ, departure is multiplied by e^{-1}.

Euler from an integral balance

$$M(t+h) - M(t) = \int_t^{t+h} (r - kM(s))\, ds.$$

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$$M_{n+1} = M_n + h(r - kM_n).$$

The quadrature uses the old removal rate.

One half-minute step

$M_n = 4$, $r = 2$, $k = .25$, $h = .5$.

$$M_{n+1} = 4 + .5(2 - .25 \cdot 4) = 4.5.$$

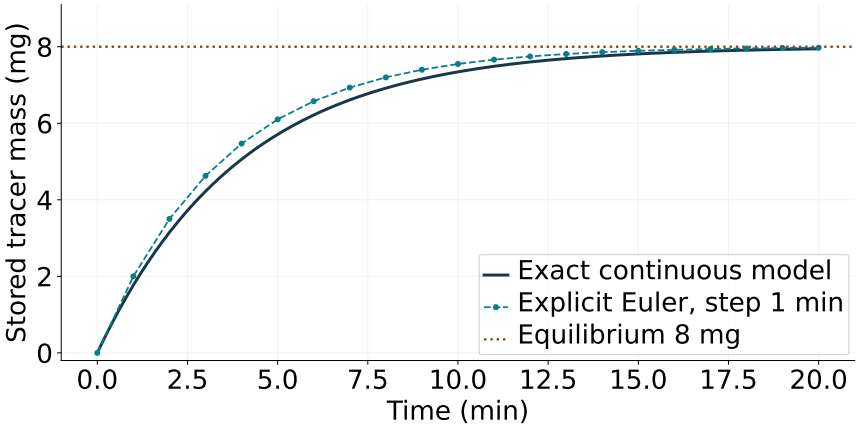
One half-minute step

$M_n = 4,\ r = 2,\ k = .25,\ h = .5.$

$$M_{n+1} = 4 + .5(2 - .25 \cdot 4) = 4.5.$$

$$M(t + h) = 8 - 4e^{-.125} \approx 4.4700.$$

Continuous and discrete filling



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$$M(t) = 8(1 - e^{-.25t}), \qquad M_n = 8(1 - .75^n) \quad (h = 1).$$

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The equilibrium agrees; finite-time trajectories differ.

Exact discrete recurrence

$$e_n = M_n - M_*, \quad e_{n+1} = (1 - kh)e_n.$$

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$$e_n = (1 - kh)^n e_0.$$

A negative multiplier alternates the side of equilibrium.

Stability interval

$$|1 - kh| < 1 \iff 0 < kh < 2.$$

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The subinterval $1 < kh < 2$ is stable but oscillatory.

Positivity of the Euler update

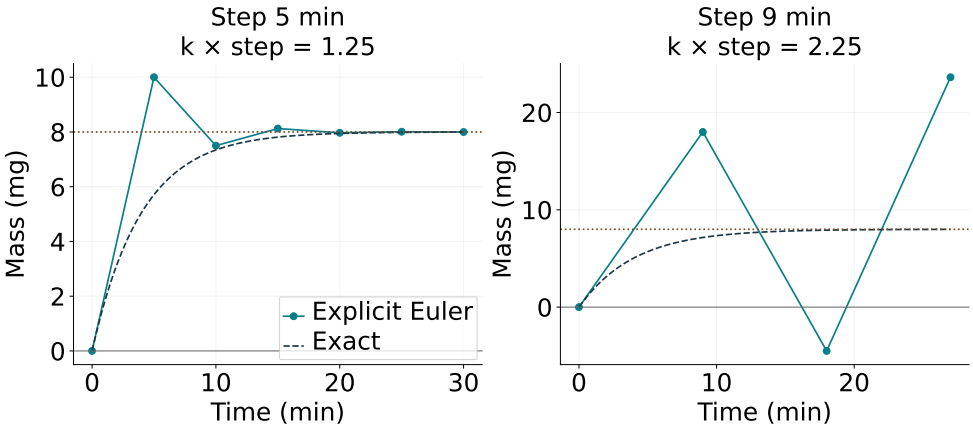
$$M_{n+1} = (1 - kh)M_n + hr.$$

Positivity of the Euler update

$$M_{n+1} = (1 - kh)M_n + hr.$$

For $M_n, r \geq 0$, $0 \leq kh \leq 1$ guarantees a nonnegative result.

Large-step failure modes



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$h = 5$: $a = -.25$, sequence $0, 10, 7.5, 8.125$.

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$h = 5$: $a = -.25$, sequence $0, 10, 7.5, 8.125$.
$h = 9$: $a = -1.25$, sequence $0, 18, -4.5, 23.625$.

Fixed-time refinement

$$E(h) = |M_h(T) - M(T)|, \qquad E(h)/E(h/2) \to 2.$$

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Match final time, initial condition, and physical parameters.

Zero removal and small-k evaluation

$k = 0$ gives $M = M_0 + rt$.

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Near zero evaluate $M_0 e^{-kt} + r[-\operatorname{expm1}(-kt)/k]$, branching at exactly zero.

Discrete mass budget

$$D_{n+1} = D_n + hkM_n \quad \Rightarrow \quad M_n + D_n = M_0 + rt_n.$$

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The identity can hold even for negative computed mass.

Numerical and physical discrepancies

Compare first with the exact equation solution, then with independent observations.

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A small numerical error does not establish the mixing or proportional-removal assumptions.