Mathematical Modeling Lecture 4

Pulsatile circulation: changing compliance and self-consistent valves

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Fall 2024

Conservation, storage, and link laws

$$\dot{\mathbf{V}} = B\mathbf{Q}, \qquad V_i = V_{d,i} + C_i(t)P_i.$$

$$Q_{\text{resistor}} = G\Delta P, \qquad Q_{\text{valve}} = G[\Delta P]_+.$$

Dots use time τ in minutes; displayed seconds satisfy $t = 60\tau$.

Topology of the closed circulation

PV $\to$ LV $\to$ SA $\to$ SV $\to$ RV $\to$ PA $\to$ PV.
Four valves are directed; two tissue resistances are bidirectional. Every transfer has two endpoint signs.

Hydraulic time units

$$[C] = \text{L/mmHg}, \quad [Q] = \text{L/min}.$$

$$10 \frac{\text{L}}{\text{min}} \frac{.06 \text{ s}}{60 \text{ s/min}} = .010 \text{ L}.$$

Shared ventricular phase

$$\phi(t) = ft/60, \qquad T = 60/f = .75 \text{ s} \quad (f = 80).$$

One phase drives both ventricular compliance functions. The baseline has no rate feedback.

Product differentiation in the storage law

$$V = V_d + CP \quad \Rightarrow \quad \dot{V} = C\dot{P} + \dot{C}P.$$

At fixed volume, CP is constant.

Isovolumic pressure change

$$V - V_d = .10 \text{ L}, \quad C = .002 \text{ L/mmHg} \Rightarrow P = 50 \text{ mmHg}.$$

Halve C: the pressure doubles at unchanged volume.

Ideal-valve inequalities

$$Q = G[P_{\text{up}} - P_{\text{down}}]_+.$$

Open: $\Delta P \geq 0$. Closed: $\Delta P \leq 0$ and $Q = 0$. Equality gives zero flow.

Backward difference of stored volume

$$\frac{C^{n+1}\mathbf{P}^{n+1} - C^{n}\mathbf{P}^{n}}{h} = -L_S\mathbf{P}^{n+1}.$$

Each pressure multiplies its matching compliance.

Fixed-state matrix problem

$$\boxed{(C^{n+1} + hL_S)\mathbf{P}^{n+1} = C^n\mathbf{P}^n.}$$

A candidate open valve contributes to both diagonal and off-diagonal entries of L_S.

Positive definiteness

$$\mathbf{z}^T(C + hL_S)\mathbf{z} = \sum_i C_i z_i^2 + h \sum_{\{i,j\}} G_{ij}(z_i - z_j)^2 > 0.$$

Positive compliance removes the pure Laplacian nullspace.

Nested time and valve iterations

Guess states $\rightarrow$ solve $\rightarrow$ test inequalities.
Hold the old pressures, old/new compliances, and clock fixed throughout this inner iteration.

An inconsistent closed candidate

$C^{\text{old}} = (1, 1)$, $P^{\text{old}} = (2, 1)$, $C^{\text{new}} = (.5, 1)$.
Closing the valve $1 \to 2$ gives $(4, 1)$, violating $P_1 \leq P_2$.

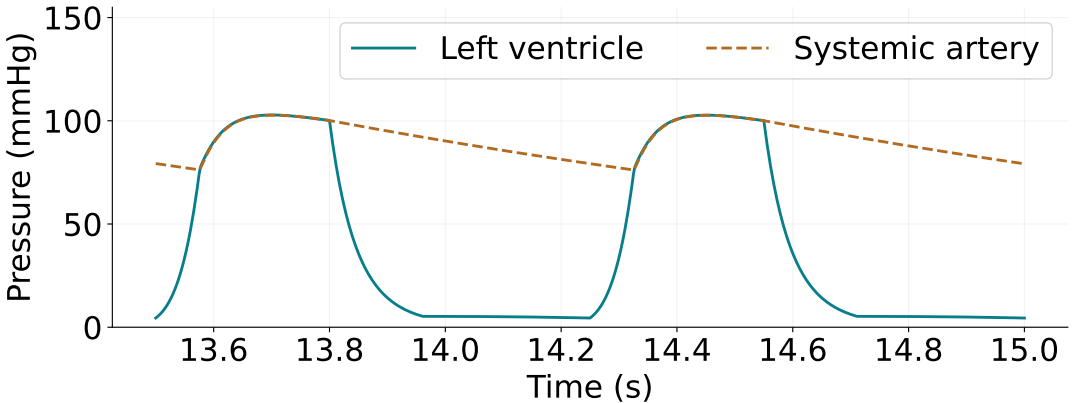
A consistent open candidate

For $hG = 1$,

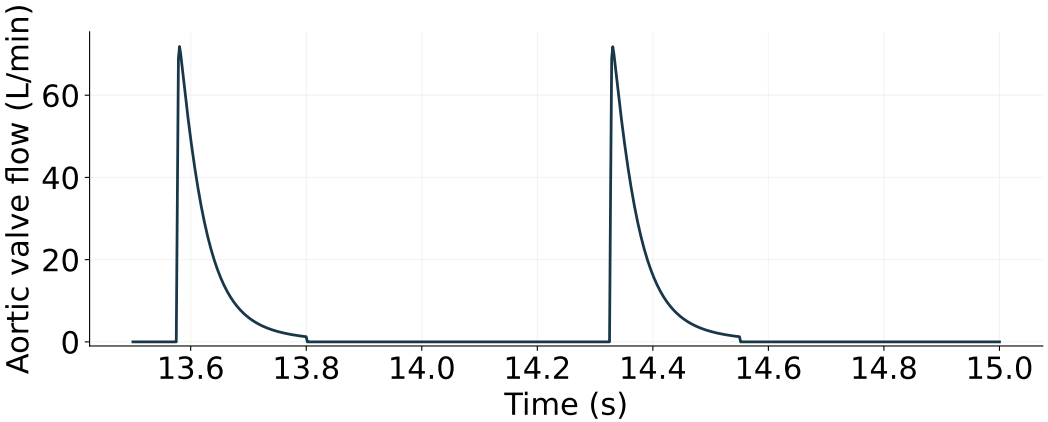
$$\begin{bmatrix} 1.5 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$$P_1 = 2.5 > P_2 = 1.75, \qquad .5P_1 + P_2 = 3.$$

Ventricular and arterial pressures



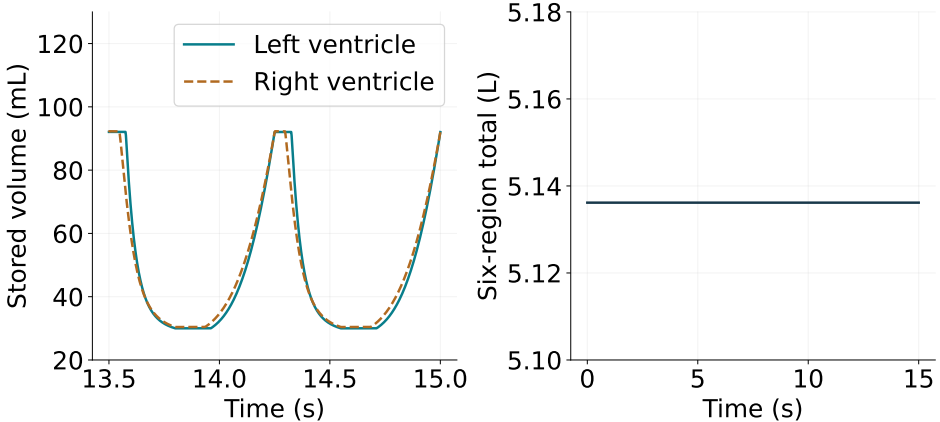
Aortic valve flow at the same times



Aortic flow and pressure crossing

$Q_{\mathrm{Ao}} > 0$ requires $P_{\mathrm{LV}} > P_{\mathrm{SA}}$.
The figure is a finite startup calculation, not a claim of exact periodicity.

Discrete total-volume conservation



Discrete total-volume conservation

$$\mathbf{1}^T L_S = 0 \Rightarrow \sum_i C_i^{n+1} P_i^{n+1} = \sum_i C_i^n P_i^n.$$

Fixed unstressed volumes complete the volume total.

Local residual and storage reconstruction

$$\mathbf{r}^{n+1} = \frac{\mathbf{V}^{n+1} - \mathbf{V}^n}{h} - B\mathbf{Q}^{n+1}.$$

Also verify valve inequalities and $V_i = V_{d,i} + C_i(\phi)P_i$.

Parameter provenance and numerical refinement

Baseline $R_s = 17.86$; reference source $R_s = 8$.
Halving the step tests time sensitivity. Changing resistance changes the physical model.
Update $G_s = 1/R_s$ consistently.