Mathematical Modeling Lecture 6

Population transfers: SIR dynamics, thresholds, and numerical conservation

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A closed population with three states

$$S + I + R = N.$$

Susceptible, infectious, and recovered are mutually exclusive categories.

Recovery retains people inside the population; it removes infectiousness, not population mass.

Homogeneous mixing and infection flux

Closed population: $S + I + R = N$



Same transfer leaves one group and enters the next

Homogeneous mixing and infection flux

$$\text{infection hazard} = \beta I/N, \qquad J_{\text{inf}} = \beta SI/N.$$

β has units day^{-1} and J_{inf} has units people/day.

Recovery as a transition hazard

$$J_{\mathrm{rec}} = \gamma I.$$

A constant hazard gives survival in the infectious state proportional to $e^{-\gamma t}$.

It does not give everyone the same recovery date.

Derivation of the SIR equations

$$\dot{S} = -J_{\text{inf}},$$
$$\dot{I} = J_{\text{inf}} - J_{\text{rec}},$$
$$\dot{R} = J_{\text{rec}}.$$

$$\dot{S} + \dot{I} + \dot{R} = 0.$$

The invariant population simplex

$$S, I, R \geq 0, \qquad S + I + R = N.$$

At an empty donor group, its outflow vanishes.

S cannot increase and R cannot decrease.

Per-infectious growth rate

$$\dot{I} = I(\beta S/N - \gamma).$$

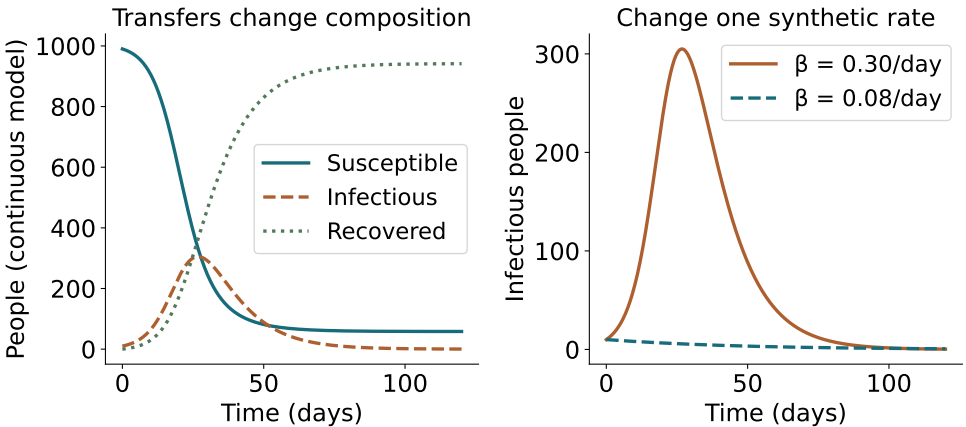
$$\mathcal{R}_{\text{eff}} = \frac{\beta}{\gamma}\frac{S}{N}; \qquad \dot{I} > 0 \iff \mathcal{R}_{\text{eff}} > 1 \quad (I > 0).$$

Initial growth in the synthetic example

$$\beta = 0.3, \quad \gamma = 0.1 \text{ day}^{-1}, \quad (S, I, R) = (990, 10, 0).$$

$$J_{\text{inf}} = 2.97, \quad J_{\text{rec}} = 1, \quad \dot{I} = 1.97 \text{ people/day}.$$

The peak is a balance of flows



The peak is a balance of flows

$$S_p = N\gamma/\beta.$$

As S falls, infection recruitment eventually falls below recovery.

The peak can occur with many susceptible people remaining.

Eliminating time from two equations

$$\frac{\mathrm{d}I}{\mathrm{d}S} = \frac{\dot{I}}{\dot{S}} = -1 + \frac{\gamma N}{\beta S}.$$

Valid on intervals with $S, I > 0$ and $\beta > 0$.

An analytic orbit relation

$$I - I_0 = -(S - S_0) + \frac{\gamma N}{\beta} \log \frac{S}{S_0}.$$

The logarithm acts on a dimensionless ratio.

This relation checks the trajectory independently of its time parametrization.

Final size and remaining susceptibility

$$\log \frac{S_\infty}{S_0} = -\frac{\beta}{\gamma N}(N - S_\infty - R_0).$$

Assume $I_\infty = 0$ and constant positive parameters.

Solve for the trajectory-consistent root; the relation does not force $S_\infty = 0$.

One conservative numerical step

$$
\begin{aligned}
S^{+} &= S - hJ_{\mathrm{inf}}, \\
I^{+} &= I + h(J_{\mathrm{inf}} - J_{\mathrm{rec}}), \\
R^{+} &= R + hJ_{\mathrm{rec}}.
\end{aligned}
$$

Compute each flux once and use it in both affected equations.

The one-day hand calculation

$$(990, 10, 0) \longmapsto (987.03, 11.97, 1).$$

$$987.03 + 11.97 + 1 = 1000.$$

Two half-steps need not give the same endpoint because their second flux evaluation uses a changed state.

A sufficient positivity restriction

$$S^+ = S(1 - h\beta I/N), \qquad I^+ = I(1 - h\gamma) + h\beta SI/N.$$

$$h \max(\beta, \gamma) \leq 1$$

ensures nonnegative states throughout the simplex.

Changing one transmission parameter

For $\beta = 0.08$, $\gamma = 0.1$ day^{-1}:

$$\beta S_0/N - \gamma = -0.0208 \text{ day}^{-1}.$$

Since S cannot increase, infectious prevalence declines from the start.

Verification of trajectories and observables

Check $S + I + R$, nonnegativity, monotonic S, R, and the analytic orbit.

Refine h at common times; compare peak height and time separately.

Conserved population does not imply an accurate epidemic curve.

Evidence for a population model

Document contact assumptions, observation process, initial conditions, and parameter units.

A continuous hazard and a discrete infection probability are different quantities.

Synthetic scenario comparison is not empirical validation.