Mathematical Modeling Lecture 7

# Finite-population genetics: inheritance, drift, and fixation

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# Diploid accounting

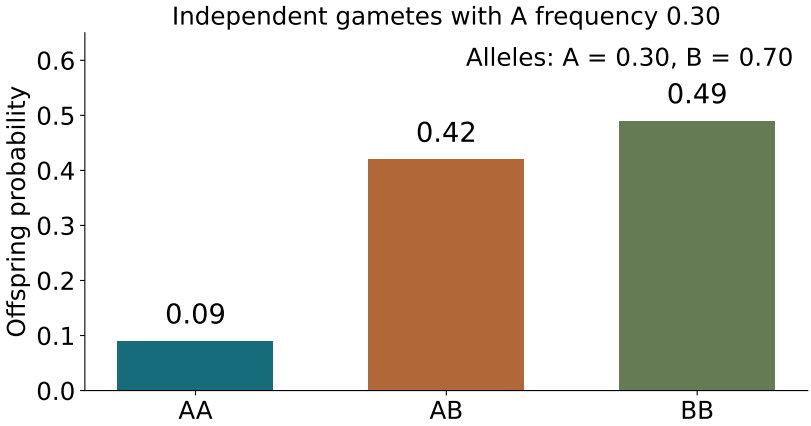
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Every heterozygote contributes one $A$ copy and one $B$ copy.

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The heterozygote combines two ordered outcomes.

# Probabilities and expected counts

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For $100$ offspring, expected counts are $(9, 42, 49)$.

A finite random sample need not equal its expected counts.

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No selection, mutation, or migration is included.

Constancy of expectation does not imply constancy of each realization.

# A finite state variable

$$K_n \in \{0, 1, \ldots, 2N\}, \qquad p_n = K_n/(2N).$$

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Offspring copies are sampled from the current parental pool.

# The binomial transition law

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$$P_{ij} = \binom{2N}{j} (i/2N)^j (1 - i/2N)^{2N-j}.$$

# One neutral generation

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A possible next count is $K_{n+1} = 18$, giving $p_{n+1} = 0.36$.

The next generation samples with probability $0.36$.

# Conditional expectation and variance

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Mean change is zero; random change need not be zero.

# Population-size scaling of fluctuations

At $p = 0.30$:

$$\sigma_{25} = \sqrt{0.21/50} \simeq 0.06481, \qquad \sigma_{250} = \sqrt{0.21/500} \simeq 0.02049.$$

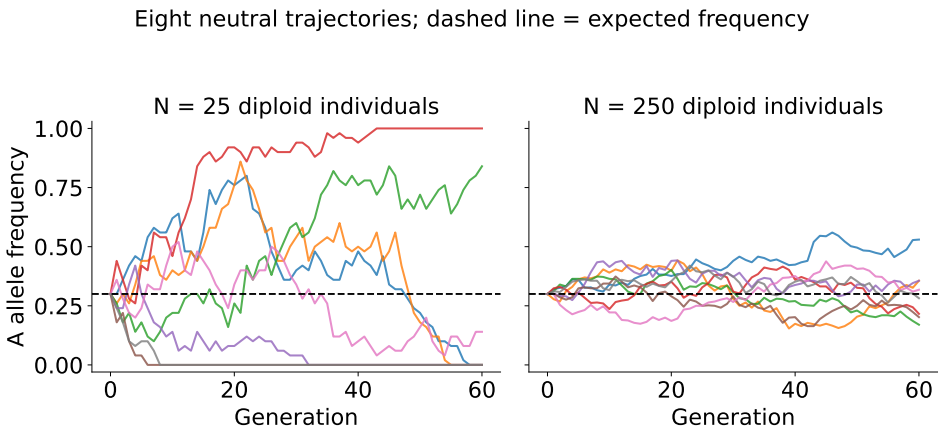
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Tenfold larger $N$ reduces standard deviation by $\sqrt{10}$.

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The plotted dashed line is $p_0$, not the mean of eight displayed runs.

# Expected heterozygosity declines

$$\mathbb{E}[p_{n+1}(1-p_{n+1}) \mid p_n] = \left(1 - \frac{1}{2N}\right) p_n(1-p_n).$$

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$$\mathbb{E}[H_n] = 2p_0(1-p_0)\left(1 - \frac{1}{2N}\right)^n.$$

# Distributional broadening

$$\mathrm{Var}(p_n) = p_0(1-p_0)\left[1-\left(1-\frac{1}{2N}\right)^n\right].$$

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The mean is fixed while probability moves toward the endpoints.

# Absorbing endpoints

$$p_n = 0 \Rightarrow p_{n+1} = 0, \qquad p_n = 1 \Rightarrow p_{n+1} = 1.$$

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Without mutation or migration, a missing allele cannot be resampled.

# Neutral fixation probability

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$$\Pr(p_\infty = 1) = \mathbb{E}[p_\infty] = p_0.$$

Fixation does not require a selective advantage.

# Computational checks for a random model

Check integer counts, the $1/(2N)$ frequency grid, bounds, and absorbing states.

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Compare repeated-draw moments with theoretical moments using sampling uncertainty.

A seed controls repeatability, not representativeness.

# Mean dynamics and individual populations

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Drift, loss, and fixation require the distributional model.