Mathematical Modeling Lecture 8

Osmotic water transport and cell volume

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An impermeant amount in a changing volume

$$n = \text{constant}, \qquad c_{\text{in}} = n/V.$$

Water movement changes concentration without changing trapped solute amount.

Concentration units

$$1\,\mathrm{fmol/pL} = \frac{10^{-15}\,\mathrm{mol}}{10^{-12}\,\mathrm{L}} = 1\,\mathrm{mM}.$$

$300\,\mathrm{fmol}$ in $1\,\mathrm{pL}$ is $300\,\mathrm{mM}$.

Osmotic and hydrostatic driving pressures

$$\Pi = \mathcal{R}Tc.$$

$$J_{\text{in}} = L_p A_m \{\Delta\Pi - \Delta P\}.$$

Greater internal osmotic pressure draws water inward; greater internal hydrostatic pressure opposes it.

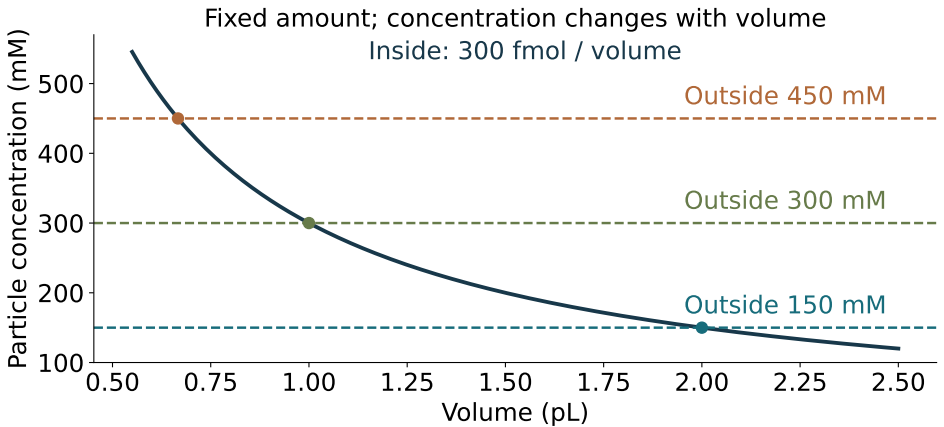
Conservation closes the scalar model

Equal hydrostatic pressures and constant effective L give

$$J_{\text{in}} = L(c_{\text{in}} - c_{\text{out}}).$$

$$\dot{V} = L\left(\frac{n}{V} - c_{\text{out}}\right).$$

The equilibrium concentration intersection



The equilibrium concentration intersection

$$n/V_* = c_{\text{out}} \quad \Longrightarrow \quad V_* = n/c_{\text{out}}.$$

For $n = 300$ and exterior $150, 300, 450\,\text{mM}$:

$$V_* = 2, 1, 2/3\,\text{pL}.$$

Phase-line analysis

$$V < V_* \Rightarrow \dot{V} > 0, \qquad V > V_* \Rightarrow \dot{V} < 0.$$

The exact solution approaches the equilibrium from its original side.

Overshoot in a computed curve may be a time-step artifact.

Local relaxation time

$$V = V_* + \eta : \qquad \dot{\eta} = -\frac{Ln}{V_*^2}\eta + O(\eta^2).$$

$$\tau = \frac{V_*^2}{Ln} = \frac{n}{Lc_{\text{out}}^2}.$$

A common dimensionless equation

$$v = V/V_*, \qquad \theta = t/\tau.$$

$$\frac{\mathrm{d}v}{\mathrm{d}\theta} = \frac{1}{v} - 1.$$

The reference volume and timescale absorb the constant parameters.

Separation of variables

$$\theta = \int_{v_0}^{v} \frac{z}{1-z}\,\mathrm{d}z.$$

$$\theta = v_0 - v + \log\left|\frac{1-v_0}{1-v}\right|.$$

This formula applies away from the constant equilibrium solution.

A decreasing stability function

$$\Phi(v) = v - 1 - \log v \geq 0.$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\theta} = -\left(1 - \frac{1}{v}\right)^2 \leq 0.$$

One signed flow calculation

$$J = 0.002(300 - 150) = 0.3\,\mathrm{pL/s}.$$

At $h = 0.2\,\mathrm{s}$:

$$V^+ = 1.06\,\mathrm{pL}, \qquad c^+ = 300/1.06 \simeq 283.019\,\mathrm{mM}.$$

Euler dynamics near equilibrium

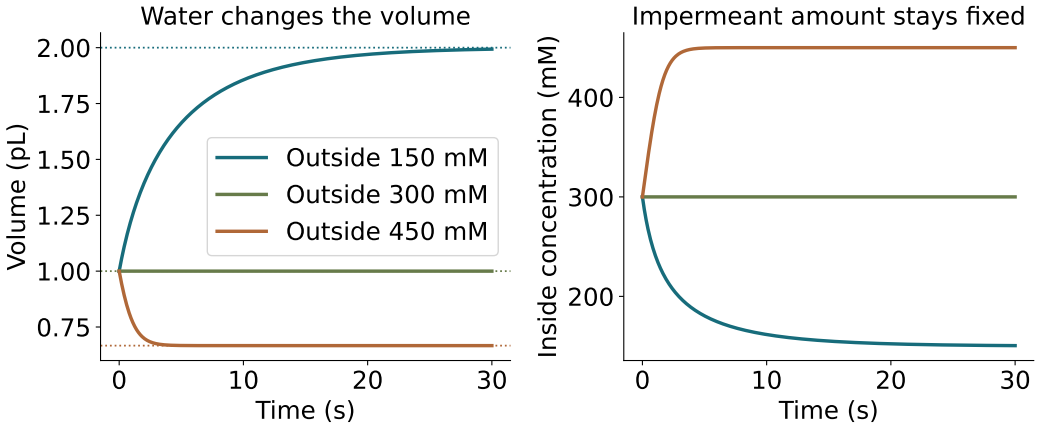
$$\eta_{n+1} \simeq (1 - h/\tau)\eta_n.$$

Stability: $0 < h/\tau < 2$.

Nonalternating local relaxation: $0 < h/\tau \leq 1$.

Both differ from an accuracy requirement.

Volume and concentration trajectories



Volume and concentration trajectories

All cases begin at $V_0 = 1\,\mathrm{pL}$ with $n = 300\,\mathrm{fmol}$.

Swelling dilutes the trapped solute; shrinking concentrates it.

The plotted step is $0.05\,\mathrm{s}$.

Zero external concentration

$$\dot{V} = Ln/V \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}V^2 = 2Ln.$$

$$V(t) = \sqrt{V_0^2 + 2Lnt}.$$

No finite equilibrium exists in this limit.

A hierarchy of numerical checks

Check $cV = n$, positive volume, and the correct direction of flow.

Compare against equilibrium, the zero-exterior exact solution, or the implicit trajectory relation.

Refine time steps at common times.

Additional mechanisms for cell-volume regulation

Permeant ions require amount balances.
Hydrostatic pressure requires mechanics.
Active transport requires pump fluxes.

The present water-only calculation isolates osmosis; it does not establish cell survival or physiological regulation.