Mathematical Modeling Lecture 10

Cable equations and propagation through a branch

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Axial Ohm law

$$I_a = -\frac{\pi a^2}{\rho}\partial_x V.$$

Cross-sectional area determines axial conduction; membrane area determines capacitive and ionic currents.

Charge conservation on a short segment

$$-\partial_x I_a = 2\pi a(C\partial_t V + I_{\text{ion}} - I_{\text{app}}).$$

Axial current lost along the segment must cross or charge its membrane.

The active cable equation

$$C\partial_t V = \frac{a}{2\rho}\partial_{xx} V - I_{\text{ion}} + I_{\text{app}}.$$

Local gate equations accompany every spatial point.

This is a reaction–diffusion system, not the classical wave equation.

Passive membrane scales

For $u = V - E_L$:

$$\tau_m u_t = \lambda^2 u_{xx} - u, \quad \tau_m = C/g_L, \quad \lambda^2 = a/(2\rho g_L).$$

Time relaxation and steady spatial attenuation are different observables.

A sealed-cable exact benchmark

$$u(x,t) = B\cos(kx)e^{-(Dk^2+g_L/C)t}, \quad k = j\pi/\ell.$$

$$D = a/(2\rho C), \qquad u_x(0,t) = u_x(\ell,t) = 0.$$

Shared edge currents

$$K_{ij} = \frac{\pi r_{ij}^2}{\rho \Delta x_{ij}}, \qquad I_{i \to j} = K_{ij}(V_i - V_j).$$

K is in mS, V in mV, and total edge current in μA.

Membrane area assigned to nodes

$$A_i = \sum_{j \sim i} \pi r_{ij} \Delta x_{ij}, \qquad M_i = C A_i.$$

Each endpoint receives half an edge lateral area.

A junction receives contributions from every incident edge.

The finite-volume voltage balance

$$M_i \dot{V}_i + A_i I_{\mathrm{ion},i} + \sum_j K_{ij}(V_i - V_j) = I_{\mathrm{app},i}.$$

Every term is a total current; membrane densities must be multiplied by area.

A charging branch junction

$$I_C = 0.006 - 0.002 - 0.001 - 0.001 = 0.002\,\mu\mathrm{A}.$$

$$\dot{V} = 0.002/0.0002 = 10\,\mathrm{mV/ms}.$$

Over $0.01\,\mathrm{ms}$, the initial-rate estimate is $0.1\,\mathrm{mV}$.

Cancellation in the global current budget

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_i M_i V_i + \sum_i A_i I_{\mathrm{ion},i} = \sum_i I_{\mathrm{app},i}.$$

Internal axial currents cancel pairwise.

Weighted charge, not unweighted voltage sum, enters the balance.

The graph Laplacian

$$(LV)_i = \sum_j K_{ij}(V_i - V_j), \qquad L\mathbf{1} = 0.$$

$$z^T L z = \sum_{\{i,j\}} K_{ij}(z_i - z_j)^2 \geq 0.$$

An implicit coupled voltage step

After the exact frozen-voltage gate update, set $G_i = \sum_j g_{ij}$ and $b_i = \sum_j g_{ij} E_j$.

$$(M/\Delta t + L + D_A \operatorname{diag}(G))\, V^{k+1} = MV^k/\Delta t + D_A b + I_{\text{app}}.$$

Positive definiteness and a unique solve

$M_i > 0$, $\Delta t > 0$, and $G_i \geq 0$ give

$$z^T(M/\Delta t + L + D_A \operatorname{diag}(G))z > 0 \quad (z \neq 0).$$

The matrix solve is uniquely defined; stability does not remove time-discretization error.

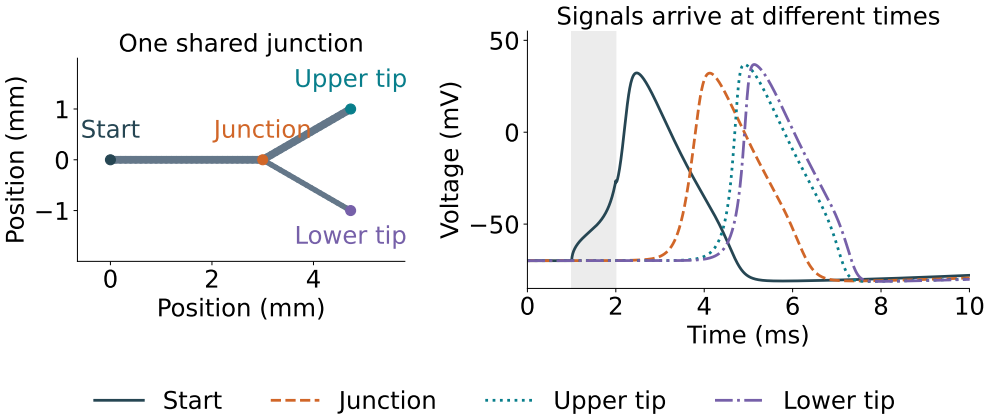
Elimination along a tree

Eliminate a leaf i attached to parent p:

$$d_p \leftarrow d_p - K_{ip}^2/d_i, \qquad r_p \leftarrow r_p + K_{ip} r_i/d_i.$$

Solve the root, then recover child voltages in reverse order.

The synthetic branched experiment

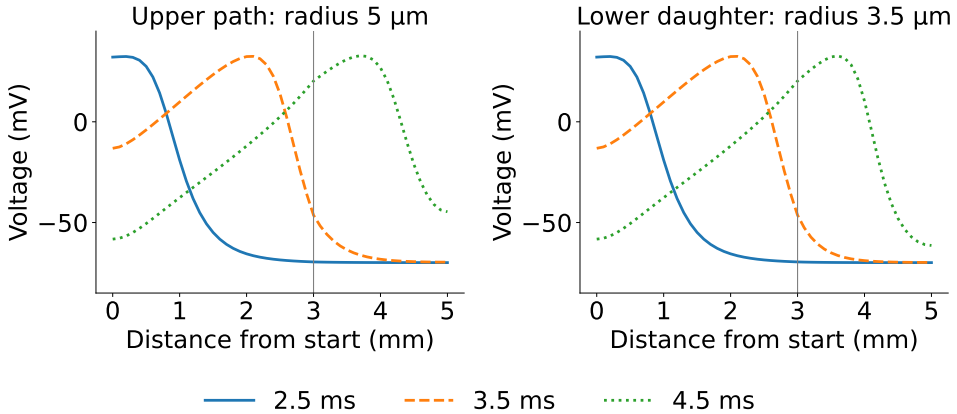


The synthetic branched experiment

Trunk $3\,\mathrm{mm}$; daughters $2\,\mathrm{mm}$ each.
Radii: $5, 5, 3.5\,\mu\mathrm{m}$.

One shared junction, 71 nodes, input $0.006\,\mu\mathrm{A}$ on $[1, 2)\,\mathrm{ms}$.

Time traces and spatial snapshots



Time traces and spatial snapshots

A trace fixes position; a snapshot fixes time.

Both path panels share the same trunk.
Snapshots: 2.5, 3.5, $4.5\,\mathrm{ms}$.

Local ionic feedback regenerates the signal as axial current spreads.

Threshold-based arrival and speed

Use the first upward $0\,\mathrm{mV}$ crossing at both probes.

$$v_{\mathrm{avg}} = \frac{2\,\mathrm{mm}}{4.692 - 3.799\,\mathrm{ms}} \simeq 2.24\,\mathrm{m/s}.$$

Arrival from stimulus onset includes initiation delay.

A discrete current-budget residual

$$\sum_i M_i \Delta V_i + \Delta t \sum_i A_i (G_i V_i^{k+1} - b_i) - \Delta t \sum_i I_{\mathrm{app},i} = 0.$$

Use the conductances and new voltages actually used by the numerical scheme.

Separate tests of algebra, space, and time

Tree solve versus dense solve: algebra.
Passive cosine versus continuum solution: space and time.
Exact discrete-system evolution: isolated time error.

Active arrival refinement at one mesh does not establish active spatial convergence.

Model evidence for a propagated signal

Geometry, stimulus, channel laws, and recording locations determine the experiment.

Measured waveforms are needed for biological validation.

The current model omits extracellular gradients and holds reversal potentials fixed.