Sampling design and Student's t

These are original, fictitious teaching datasets. State the parameter, assumptions and tail direction before calculating. Use a calculator for arithmetic. Write conclusions about population means, not individual plants or flasks.

Use the Chapter 9 notation: df is degrees of freedom, and $P(T_{df} > t_{df,\alpha}) = \alpha$. A one-sided level-$\alpha$ test uses $t_{df,\alpha}$; a two-sided level-α test or a $100(1-\alpha)\%$ confidence interval uses $t_{df,\alpha/2}$.

Problem 1. Does the nutrient treatment improve growth?

A botanist wants to study whether a new nutrient treatment increases mean stem growth compared with the standard treatment. She considers two designs:

- **Design A:** select eight independent mother plants. Take two comparable cuttings from each mother plant; these two cuttings form one pair. Randomly assign one cutting in each pair to the new treatment and the other to the standard treatment. Measure growth after the same period. Assume the eight paired growth differences are independent across mother plants and normally distributed.
- **Design B:** select sixteen independent plants, each from a different mother plant, and randomly assign eight to each treatment. Assume normally distributed growth in each group with a common, unknown population variance.

The botanist uses **Design A**. For each mother plant, subtract the growth of its standard-treatment cutting from the growth of its new-treatment cutting. Thus each difference compares the two cuttings from the same mother plant. Define D = new-treatment growth minus standard-treatment growth, in millimeters. The eight observed differences are

$$0, \quad 1, \quad 2, \quad 3, \quad 3, \quad 4, \quad 5, \quad 6.$$

Let μ_D be the population mean difference. Test $H_0 : \mu_D = 0$ against $H_a : \mu_D > 0$ at significance level $\alpha = 0.05$.

(a) Design and method. Which test—paired t or independent pooled t—belongs to each design? Give the degrees of freedom for each. Explain why the eight differences in Design A constitute the sample for its test, even though sixteen cuttings were measured. Does a paired test require equal population variances for the two treatments?

(b) Calculation and evidence. Calculate $\bar{d}$, s_D^2, the estimated standard error and the test statistic. Use the shared table on page 3 to give a bound on the one-sided p-value. State the decision and a conclusion in context.

(c) Errors and significance. Describe a Type I error and a Type II error in this setting. Explain the meaning of $\alpha = 0.05$. Does rejecting H_0 tell us that our conclusion has a 5% probability of being wrong? Explain.

(d) Power at a specified effect. A separate planning calculation for this same eight-pair, level-0.05 t test gives

$$P(\text{reject } H_0 \mid \mu_D = 2 \text{ mm}, \ \sigma_D = 2 \text{ mm}) \approx 0.815.$$

Use this supplied value; you do not need to derive it. Each repetition of the test still estimates its standard deviation from its own sample.

Find β and power at this specified alternative, and interpret both. If the botanist lowers α from 0.05 to 0.01 while keeping the sample size, true effect and population variation fixed, what happens to the rejection region, β and power? Is $\beta = 1 - \alpha$?

Problem 2. Which interval supports the conclusion?

A microbiologist compares algal dry biomass after seven days in a new growth medium and a standard medium. Each observation comes from a **different, independently grown flask**. There is no matching between the two groups. The measurements, in milligrams, are

Medium	Measurements (mg)
New	17, 19, 21, 23, 25
Standard	12, 14, 16, 17, 18, 20, 22

For this exercise, assume independent normal populations with a common, **unknown** variance σ^2. Define $\Delta = \mu_{\text{new}} - \mu_{\text{standard}}$.

(a) Build the standard error. Calculate both sample means and both sample variances using divisor $n_i - 1$. Then calculate the pooled variance s^2 (course-slide notation) and estimated standard error of the mean difference. Give the degrees of freedom and connect them to the two sample means that were estimated. Explain why pairing the first five entries would not be justified.

(b) A t interval. Construct a 95% confidence interval for Δ. Interpret its units and give an accurate meaning of 95% confidence. Use the interval to decide whether the matching two-sided test of $H_0 : \Delta = 0$ rejects at level 0.05. Verify the decision with the test statistic and a p-value bound using the shared table on page 3.

(c) A tempting shortcut. A colleague uses the same estimated standard error from (a), but replaces the t critical value from (b) by the standard-normal critical value 1.960. Compute the resulting interval. How does its width and conclusion compare with (b)? Explain why this is a **naive plug-in z interval**, rather than an exact 95% interval under the stated model. What uncertainty does the t procedure account for? How does the critical-value difference behave as the degrees of freedom increase?

(d) A different information setting. Now suppose, in a separate version of the problem, both population standard deviations really are **known** to be 3 mg. Retain the normality and independence assumptions. Write the correct known-σ z interval and its standard error. Is this procedure valid despite the small sample sizes? Explain why it differs from the shortcut in (c). Compute its endpoints if time permits.

Closing check. A statistically inconclusive comparison is not evidence that two population means are exactly equal. Explain this distinction using your answer to (b).

Student t reference table

Entries are $t_{df,\alpha}$ with $P(T_{df} > t_{df,\alpha}) = \alpha$. The first column gives df; the remaining column headings give the probability α to the right of the listed value. Values are rounded to three decimal places.

df	0.10	0.05	0.025	0.01	0.005	0.001
4	1.533	2.132	2.776	3.747	4.604	7.173
5	1.476	2.015	2.571	3.365	4.032	5.893
6	1.440	1.943	2.447	3.143	3.707	5.208
7	1.415	1.895	2.365	2.998	3.499	4.785
8	1.397	1.860	2.306	2.896	3.355	4.501
9	1.383	1.833	2.262	2.821	3.250	4.297
10	1.372	1.812	2.228	2.764	3.169	4.144
12	1.356	1.782	2.179	2.681	3.055	3.930
14	1.345	1.761	2.145	2.624	2.977	3.787
20	1.325	1.725	2.086	2.528	2.845	3.552

For each problem, determine the appropriate degrees of freedom and tail probability from the design and the requested inference. Record the row and column you use.

Show calculations and explanations on separate paper.